Stable reduction of algebraic curves

Lectured by
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Contents

1. Moduli space and stable curves

1.1. **Motivations.** The main topic of this mini-course is stable reduction of algebraic curves. Before that, we need to know what is a "stable curve". The motivations of stable curves origin from the study of moduli spaces. Let M_g be the (coarse) moduli space of projective smooth curves of genus g over an algebraically closed field k, which is an algebraic variety over k. It turns out that M_g is not projective (even not proper), so we want to "compactify" M_g, that is, find an open immersion

$$M_g \hookrightarrow \overline{M}_g,$$

where $\overline{M}_g$ is a proper algebraic variety over k. There are many different compactifications of M_g, such as Deligne-Mumford compactification and Satake compactification. The stable curves appears as the boundary divisors of the Deligne-Mumford compactification.

The second topic of this mini-course is degeneration of algebraic curves. A *family of algebraic varieties* is a projective flat morphism $f: X \to S$ such that there exists an open dense subset $V \subseteq S$ such that $X \times_S V \to V$ is smooth. If there exists a closed point $s_0 \in S$ such that X_{s_0} is singular, we say that $X \to S$ *degenerates* at s_0.

The degeneration plays an important role in arithmetic geometry: Let C be a smooth projective curve over $\mathbb{Q}$ and extend C to a scheme $\mathcal{C}$ over $\mathbb{Z}$, that is, the generic fiber of $\mathcal{C}$ is isomorphic to C. A natural question is, does there exist a prime number p such that

$$\mathcal{C} \times_{\mathbb{Z}} \mathbb{F}_p \to \operatorname{Spec} \mathbb{F}_p$$

is a smooth projective curve over $\mathbb{F}_p$? If the answer is yes, then it is called a *good reduction*, otherwise it is called a *degeneration* at p.

Theorem 1.1.1 (Abhyankar, Fontaine)**.** If $A \to \operatorname{Spec} \mathbb{Z}$ is an abelian scheme, then $A_{\mathbb{F}_p}$ is of dimension zero.

Corollary 1.1.1. If C is smooth projective curve over $\mathbb{Q}$ of genus $g \geq 1$. Then C degenerates at at least one prime p.

Remark 1.1.1. This is false if $\mathbb{Q}$ is replaced by a general number field.

Theorem 1.1.2 (Shafarevich)**.**

(1) If $f: X \to \mathbb{P}^1_k$ is a family of smooth projective curves, then f is isotrivial[1].
(2) If $f: X \to E$ is a family of abelian schemes over an elliptic curve E, then f is isotrivial.

1.2. **Moduli space of smooth projective curves of genus g.** Let k be an algebraically closed field. We denote the category of algebraic varieties[2] over k by Var_k and denote the category of sets by Set. A *functor of points* is a

[1] A morphism is called *isotrivial*, if all of fibers are isomorphic to each other.

[2] An *algebraic variety* over k is a scheme of finite type over k.

contravariant functor $F\colon \mathrm{Var}_k \to \mathrm{Set}$. For any algebraic variety M over k, it defines a special functor

$$\begin{aligned} h_M\colon \mathrm{Var}_k &\to \mathrm{Set} \\ X &\mapsto \mathrm{Hom}_{k-\mathrm{Var}}(X, M), \end{aligned}$$

where $\mathrm{Hom}_{k-\mathrm{Var}}(X, M)$ denotes the set of all algebraic morphisms between X and M. A functor of points F is called *representable*, if there exists a natural isomorphism $F \cong h_M$ for some algebraic variety M.

Consider the functor

$$\begin{aligned} \mathcal{M}_g\colon \mathrm{Var}_k &\to \mathrm{Set} \\ X &\mapsto \{\text{families of smooth projective curves of genus } g \text{ over } X\}/\sim. \end{aligned}$$

A natural question is whether $\mathcal{M}_g$ representable or not. The answer is no. If $\mathcal{M}_g$ is represented by an algebraic variety M_g, that is, there exists a natural isomorphism $\eta\colon \mathcal{M}_g \to h_{M_g}$.

By the natural isomorphism η, we can define the *universal curve* $\mathcal{C}_g \to M_g$ by $\eta^{-1}_{M_g}(\mathrm{id}_{M_g}) \in \mathcal{M}_g(M_g)$, which satisfies the following universal property: For any $S \to M_g \in \mathcal{M}_g(S)$, there exists a unique $X \to S$ such that the following diagram is Cartesian

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{C}_g \\ \downarrow & & \downarrow \\ S & \longrightarrow & M_g. \end{array}$$

Let $C \in \mathcal{M}_g(\mathrm{Spec}\, k)$ be a smooth projective curve over k such that there exists a non-trivial automorphism $\sigma\colon C \to C$. Then we can construct a non-trivial family $X \to S$ such that every fiber is isomorphic to C. In other words, $S \to M_g$ is the constant map. But in this case, the Cartesian diagram is

$$\begin{array}{ccc} S \times C & \longrightarrow & \mathcal{C}_g \\ \downarrow & & \downarrow \\ S & \longrightarrow & M_g, \end{array}$$

which contradicts to $X \ncong S \times C$. This shows that existence of universal family is a strong constraint, and it can be violated by existence of automorphisms.

Remark 1.2.1. Although the functor of smooth projective curves of genus g is not representable, the moduli functor of smooth projective curves with level structure[3] is represented by an algebraic variety H_g, and there exists a finite surjective map $H_g \to M_g$.

[3] Fix $N \in \mathbb{Z}_{\geq 2}$, which is prime to $\mathrm{char}(k)$, a *level structure* on smooth projective curve C with genus g is an isomorphism

$$\mathrm{Jac}(C)[N] \to (\mathbb{Z}/N\mathbb{Z})^{2g}.$$

Instead of considering representability, we should consider coarse moduli space. A *coarse moduli space* is an algebraic variety M_g over k with a natural transformation $\mathcal{M}_g \to h_{M_g}$, such that

(1) $\mathcal{M}_g(\Omega) = h_{M_g}(\Omega)$ for any algebraically closed field Ω.
(2) For any algebraic variety M' with a natural transformation $\mathcal{M}_g \to h_{M'}$ satisfies (1), there exists a morphism $M_g \to M'$ such that

$$\begin{tikzcd}
\mathcal{M}_g \arrow[rr] \arrow[dr] & & h_{M'} \\
& h_{M_g}. \arrow[ur] &
\end{tikzcd}$$

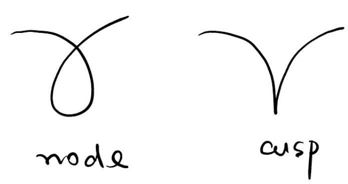
Proposition 1.2.1.

(1) M_g exists;
(2) M_g is normal and irreducible;
(3) $\dim_k M_g = 3g - 3$.

1.3. **Stable curves.**

1.3.1. *Nodal curve.* Let k be an algebraically closed field and C be a projective reduced curve over k with the normalization $\pi\colon \widetilde{C} \to C$. For $\mathcal{O}_C$-module $\pi_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C$, it only supports at the singular locus C_{Sing}, which is a finite set.

For each $p \in C_{\mathrm{Sing}}$, the stalk $(\pi_*\mathcal{O}_{\widetilde{C}})_p$ is a finite dimensional k-vector space, and the *multiplicity* δ_p is defined as $\dim_k(\pi_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)_p$. The multiplicity one case can be considered as the simplest singularity. There are two kinds of multiplicity one singularity: node and cusp.



Definition 1.3.1. A point $p \in C$ is called a *node*, if $\delta_p = 1$ and $|\pi^{-1}(p)| = 2$.

Proposition 1.3.1 ([Liu02, Proposition 7.5.15])**.** The following statements are equivalent:

(1) $p \in C$ is a node;
(2) $\pi^{-1}(p) = \{p_1, p_2\}$, and locally at $p \in V$, we have

$$\mathcal{O}_C(V) = \{f \in \mathcal{O}_{\widetilde{C}}(\pi^{-1}(V)) \mid f(p_1) = f(p_2)\}.$$

(3) The formal completion $\widehat{\mathcal{O}}_{C,p} \cong k[[u,v]]/(uv)$.

Proof. For simplicity, locally around $p \in V \subseteq C$, we fix the following notations:

$$\begin{aligned} A &= \mathcal{O}_C(V), \\ A_1 &= \{f \in \mathcal{O}_{\widetilde{C}}(\pi^{-1}(V)) \mid f(p_1) = f(p_2)\}, \\ B &= \mathcal{O}_{\widetilde{C}}(\pi^{-1}(V)). \end{aligned}$$

(1) $\Longleftrightarrow$ (2): The assumption in (2) implies that $A = A_1$. Since we have the following exact sequence

$$0 \to A_1 \to B \to k \to 0,$$

where the map $B \to k$ is defined by $f(\widetilde{p}_1) - f(\widetilde{p}_2)$, then $\dim_k B/A = \dim_k B/A_1 = 1$, that is, $\delta_p = 1$. Conversely, if p is a node, then $\dim_k B/A = \dim_k B/A_1 = 1$, and thus $A = A_1$. □

Lemma 1.3.1. Let $p \in C$ be a node. Then

(1) $\dim T_{C,p} = 2$;
(2) $\operatorname{ann}_{\mathcal{O}_{C,p}}\left((\pi_*\mathcal{O}_{\widetilde{C}})_p/\mathcal{O}_{C,p}\right) = \mathfrak{m}_p$.

Proof. For (1). Use the structure of $\widehat{\mathcal{O}}_{C,p}$. For (2). Since $\operatorname{ann}_A B/A$ is an ideal of A, and it is not equal to A, otherwise $A = B$ and p is a smooth point, so $\operatorname{ann}_A B/A \subseteq \mathfrak{m}_p$. Conversely, □

1.3.2. *Stable curve.*

Definition 1.3.2. Let C be an algebraic curve over an algebraically closed field k.

(1) C is called *semi-stable*, if it is reduced and nodal[4].
(2) C is called *stable*, if it is semi-stable, and the following conditions are verified:
  (a) C is connected, projective, with arithmetic genus[5] $p_a(C) \geq 2$;
  (b) For all irreducible component $\Gamma \subset C$, if $\Gamma \cong \mathbb{P}^1_k$, then $\Gamma \cap \overline{C \setminus \Gamma}$ contains at least three point.

Remark 1.3.1. The third condition is called the *stability condition*. Later we will show that

(1) If C is a stable curve, then $\operatorname{Aut}(C)$ is finite.
(2) Let ω_C be the dualizing sheaf of C. Then $\omega_C^{\otimes n}$ is very ample for $n \geq 3$.

If we do not require the stability condition, then $\operatorname{Aut}(C)$ is infinite and ω_C is not ample. Roughly speaking, the finiteness of automorphism group is related to the fact that any automorphism of $\mathbb{P}^1_k$ with three points fixed is identity.

Lemma 1.3.2. Let C be a reduced connected projective curve over a field k. Then

$$p_a(C) = 1 - n + \sum_{p \in C_{\mathrm{Sing}}} \delta_p + \sum_{i=1}^{n} p_a(\widetilde{\Gamma}_i),$$

[4]A curve is *nodal*, if every singular point is a node.

[5]Let C be a projective curve over a field k. The *arithmetic genus* is defined as $1 - \chi_k(\mathcal{O}_C)$.

where $C = \bigcup_{i=1}^{n} \Gamma_i$ is the irreducible decomposition of C, and $\widetilde{\Gamma}_i$ is the normalization of Γ_i.

Proof. The short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \pi_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C \to 0$$

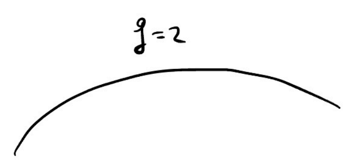
gives the long exact sequence

$$0 \to k \to H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \to \bigoplus_{p \in C_{\mathrm{Sing}}} (\mathcal{O}_{\widetilde{C}} / \mathcal{O}_C)_p \to H^1(C, \mathcal{O}_C) \to H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \to 0.$$

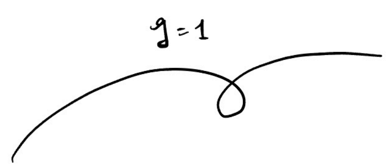
This completes the proof. $\square$

Example 1.3.1. *For $g = 2$ case, stable curve is of one of the following types:*

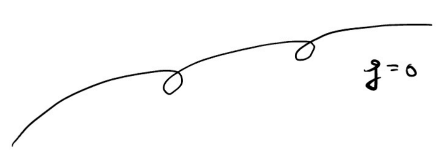
Type (a):



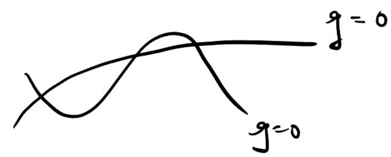
Type (b): $y^2 = x^2(x^3 + a)$, where $a \in k^*$ and $char(k) \neq 2$.



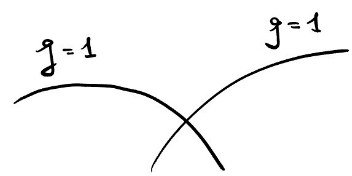
Type (c): $y^2 = x^2(x+1)^2(x+2)$.



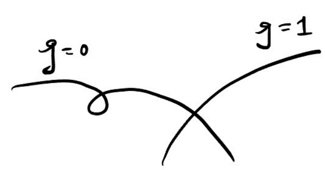
Type (d): $y^2 = (x(x+1)(x+2))^2$.



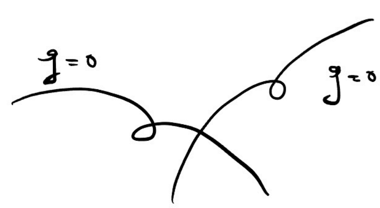
Type (e):



Type (f):



Type (g):



Remark 1.3.2. Let M_2 be the coarse moduli space of smooth projective curves of genus two and $\overline{M}_2$ be the Deligne-Mumford compactification. Then $\overline{M}_2$ is of dimension three and thus $\overline{M}_2 \setminus M_2$ is of dimension two. Moreover, $\overline{M}_2 \setminus M_2$ two irreducible components Δ_0 and Δ_1, where Δ_0 consists of stable curves of type (b),(c) and (d).

1.3.3. *Very ampleness of dualizing sheaf.*

Proposition 1.3.2. Let C be a semi-stable curve over an algebraically closed field k.

(1) C is of locally complete intersection.
(2) Let $\pi : \widetilde{C} \to C$ be the normalization. Then $\pi^* \omega_C = \omega_{\widetilde{C}}(\pi^{-1}(C_{\mathrm{Sing}}))$.
(3) For any $\Gamma \subset C$ be an irreducible component. Then

$$\omega_{C/k}|_\Gamma = \omega_{\Gamma/k}\left(\Gamma \cap \overline{C \setminus \Gamma}\right).$$

Proof. For (1). See (d) of [Liu02, Lemma 10.3.7].
For (2). See (b) of [Liu02, Lemma 10.3.12].
For (3). See proof of [Liu02, Corollary 10.3.13]. □

Corollary 1.3.1. Let C be a semi-stable curve over an algebraically closed field k. Let $\Gamma \subset C$ be an irreducible component. Then

$$\deg(\omega_{C/k}|_\Gamma) = 2(p_a(\Gamma) - 1) + \left|\Gamma \cap \overline{(C \setminus \Gamma)}\right|.$$

Corollary 1.3.2. Let C be a semi-stable curve an algebraically closed field k. If C is connected and $p_a(C) \geq 2$, then C is stable if and only if $\deg \omega_{C/k}|_\Gamma > 0$ for all irreducible component $\Gamma \subset C$.

Theorem 1.3.1. Let C be a stable curve over an algebraically closed field k. Then

(1) $H^1(C, \omega_C^{\otimes n}) = 0$ for all $n \geq 2$;
(2) $\omega_C^{\otimes n}$ is very ample for $n \geq 3$.

Proof. For (1). By duality we have

$$H^1(C, \omega_C^{\otimes n})^\vee = H^0(C, \omega_C^{\otimes(1-n)}).$$

For any irreducible component Γ, we have

$$\deg\left(\omega_C^{\otimes(1-n)}|_\Gamma\right) = (1-n)\deg \omega_C|_\Gamma < 0.$$

Thus $H^0(\Gamma, \omega_C^{\otimes 1-n}\big|_\Gamma) = 0$ since Γ is integral. Note that we have the following exact sequence

$$0 \to \omega_C^{\otimes 1-n} \to \bigoplus_\Gamma \omega_C^{\otimes 1-n}|_\Gamma,$$

so $H^0(\Gamma, \omega_C^{\otimes 1-n}\big|_\Gamma) = 0$ for every irreducible component implies $H^0(\Gamma, \omega_C^{\otimes 1-n}) = 0$.

For (2). See Corollary of Theorem 1.2 in [DM69]. □

Corollary 1.3.3. Let $f : X \to S$ be a relative stable curve. Then

(1) $R^1 f_* \omega_{X/S}^{\otimes n} = 0$ for $n \geq 2$;
(2) $\omega_{X/S}^{\otimes n}$ is relatively very ample for $n \geq 3$.

1.3.4. *Automorphism of stable curves.* Let $(X, \mathcal{L})$ be a polarized projective variety over a field k, where $\mathcal{L}$ is a very ample divisor on X. Then we define

$$\mathrm{Aut}(X, \mathcal{L}) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma^* \mathcal{L} \cong \mathcal{L}\}.$$

Lemma 1.3.3. $\mathrm{Aut}(X, \mathcal{L})$ is an algebraic group.

Proof. Since $\mathcal{L}$ is very ample, it gives an inclusion $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L}))$. Then for any $\sigma \in \mathrm{Aut}(X, \mathcal{L})$, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\sigma"'] & \mathbb{P}(H^0(X, \mathcal{L})) \arrow[d, "\sigma"] \\ X \arrow[r, hook] & \mathbb{P}(H^0(X, \mathcal{L})). \end{tikzcd}$$

This shows that $\mathrm{Aut}(X, \mathcal{L}) \hookrightarrow \mathrm{Aut}(\mathbb{P}(H^0(X, \mathcal{L}))) = \mathrm{PGL}(H^0(X, \mathcal{L}))$ is a subgroup, and it remains to show $\mathrm{Aut}(X, \mathcal{L})$ is a closed subvariety of $\mathrm{PGL}(H^0(X, \mathcal{L}))$.

For $\sigma \in \mathrm{PGL}(H^0(X, \mathcal{L}))$, we have $\sigma \in \mathrm{Aut}(X, \mathcal{L})$ if and only if $\sigma(X) = X$. Suppose $X = V(I)$, where I is a homogeneous ideal. Then $\sigma(X) = X$ is equivalent to say for any $x \in X$ and $G \in I$, we have $G(\sigma(x)) = 0$, which is a closed condition. This completes the proof. □

Proposition 1.3.3. Let $G = \mathrm{Aut}(X, \mathcal{L})$. Then

$$\dim T_{G, \mathrm{id}_X} = \dim \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X).$$

Proof. For $\sigma \in G$, any element in $T_{G,\sigma}$ is equivalent to a morphism $\tau : X_{k[\epsilon]/(\epsilon^2)} \to X_{k[\epsilon]/(\epsilon^2)}$ which lifts $\sigma : X \to X$, that is, the following diagram commutes:

$$\begin{tikzcd} X_{k[\epsilon]/(\epsilon^2)} \arrow[r, "\tau"] \arrow[d] & X_{k[\epsilon]/(\epsilon^2)} \arrow[d] \\ X \arrow[r, "\sigma"] & X. \end{tikzcd}$$

Since the underlying topological spaces of X and $X_{k[\epsilon]/(\epsilon^2)}$ are the same, we have τ is determined by a morphism

$$\begin{aligned} \mathcal{O}_X &\to \sigma_* \mathcal{O}_{X_{k[\epsilon]/(\epsilon^2)}} = \sigma_* \mathcal{O}_X \oplus \epsilon \sigma_* \mathcal{O}_X \\ a &\mapsto \sigma^\sharp(a) + \epsilon \varphi(a). \end{aligned}$$

The condition for φ to be a k-linear homomorphism implies

$$\varphi(a_1 a_2) = \sigma^\sharp(a_2)\varphi(a_1) + \sigma^\sharp(a_1)\varphi(a_2).$$

If we take $\sigma = \mathrm{id}_X$, then it gives

$$\varphi(a_1 a_2) = a_1 \varphi(a_2) + a_2 \varphi(a_1),$$

that is, $\varphi \in \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$. □

Corollary 1.3.4. If C is a stable curve and $\mathcal{L} = \omega_C^{\otimes 3}$, then

$$\mathrm{Aut}(C) := \mathrm{Aut}(C, \mathcal{L})$$

is an étale algebraic group[6].

Proof. It suffices to prove

$$\mathrm{Hom}_{\mathcal{O}_C}(\Omega^1_{C/k}, \mathcal{O}_C) = 0.$$

Let $\pi\colon \widetilde{C} \to C$ be the normalization. Then we claim

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_C}(\Omega^1_{C/k}, \mathcal{O}_X) &\cong \mathrm{Hom}_{\mathcal{O}_C}(\pi_*\mathcal{O}_{\widetilde{C}}, \mathcal{O}_C)\\ &\cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega^1_{\widetilde{C}/k}, \mathcal{O}_{\widetilde{C}}(-\pi^{-1}(C_{\mathrm{Sing}}))).\end{aligned}$$

Proof of claim. □

□

[6]An algebraic group is étale if it is smooth and finite.

2. Stable reduction

2.1. **Model.** A Noetherian, connected regular scheme of dimension[7] 1 or 0 is called a *Dedekind scheme*. Let S be a Dedekind scheme. Then for any affine open subset $U \subset S$, we have $\mathcal{O}_S(U)$ is a Dedekind domain or a field.

Example 2.1.1.

(1) *If S is a smooth connected curve over a field k, then S is a Dedekind scheme with function field $K = K(S)$.*
(2) *Let R be a discrete valuation ring. Then $S = \operatorname{Spec} R$ is a Dedekind scheme with function field $K = \operatorname{Frac}(R)$.*
(3) *Let K be a number field and $\mathcal{O}_K$ be the ring of algebraic integers in K. Then $S = \operatorname{Spec} \mathcal{O}_K$ is a Dedekind scheme with function field K.*

Definition 2.1.1. Let S be a Dedekind scheme and K be the function field of S. Let X be an algebraic variety over K.

(1) A *model* of X over S is a S-scheme $\mathfrak{X} \to S$ together with an K-isomorphism $\mathfrak{X}_K \cong X$.
(2) A morphism $\mathfrak{X} \to \mathfrak{X}'$ of two models of X is a morphism of S-shechmes that is compatible with the isomorphisms $\mathfrak{X}_K \cong X$ and $\mathfrak{X}'_K \cong X$;
(3) A model $\mathfrak{X} \to S$ verifies a property (P) if $\mathfrak{X} \to S$ verifies (P).

Example 2.1.2. *Let C be a projective curve over a field K, defined by homogeneous polynomials $F_1, \dots, F_m \in K[T_0, \dots, T_n]$. Suppose $A \subset K$ is a subring such that all of F_i have coefficients in A and $\operatorname{Frac}(A) = K$. Then the scheme*

$$\mathcal{C} := \operatorname{Proj} A[T_0, \dots, T_n]/(F_1, \dots, F_m)$$

is a model of C over $S = \operatorname{Spec} A$, since its generic fiber

$$\begin{aligned}\mathcal{C}_\eta &= \operatorname{Proj} A[T_0, \dots, T_n]/(F_1, \dots, F_m) \times_{\operatorname{Spec} A} \operatorname{Spec} K \\ &\cong \operatorname{Proj} K[T_0, \dots, T_n]/(F_1, \dots, F_m).\end{aligned}$$

Example 2.1.3. *Let C be the projective curve over $\mathbb{Q}$ defined by the equation*

$$x^q + y^q + z^q = 0.$$

Let $\mathcal{C}$ be the closed subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ defined by the same equation. Then $\mathcal{C} \to \operatorname{Spec} \mathbb{Z}$ is a model of C over $\operatorname{Spec} \mathbb{Z}$.

If X is an algebraic variety with certain properties (projective, normal, smooth), we would of course also like to find a model which preserves as many as properties of X as possible.

Proposition 2.1.1. Let X be a projective variety over a field K.

(1) There exists flat projective model.
(2) If X is normal and S is excellent[8], then there exists flat normal model.

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[7] Usually a Dedekind domain has dimension 1. Here we admit the dimension 0 case because we want to make the class of Dedekind scheme stable by localization.

[8] An excellent scheme is a complicated condition, here we provide some examples: Any algebraic variety over a field is excellent; Any Dedekind domain of characteristic zero is excellent.

On the other hand, if C is a smooth projective curve, then we cannot find a smooth model when $g(C) \geq 1$, since by Corollary 1.1.1 there always some prime p such that $\mathcal{C}_p \to \operatorname{Spec} \mathbb{F}_p$ is not smooth. But for smooth projective line over $\mathbb{Q}$, it admits infinite many smooth models.

Example 2.1.4. *Let $X = \mathbb{P}^1_{\mathbb{Q}}$ be the projective line over $\mathbb{Q}$. Then $\mathbb{P}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ together with an isomorphism of $\mathbb{P}^1_{\mathbb{Q}}$ is a model of $\mathbb{P}^1_{\mathbb{Q}}$. The morphism defined by $\times 2$ is a morphism between models, that is, the following diagram commutes*

$$\begin{tikzcd} x \in \mathbb{P}^1_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{Q}} \ni 2x \arrow[d] \\ \mathbb{P}^1_{\mathbb{Z}} \arrow[r] & \mathbb{P}^1_{\mathbb{Z}}. \end{tikzcd}$$

However, it is not an isomorphism $\mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$. In other words, $\mathbb{P}^1_{\mathbb{Q}}$ has infinite many smooth models over $\operatorname{Spec} \mathbb{Z}$.

Remark 2.1.1. If X is an abelian variety, although there may not exist smooth projective model, there always exists a smooth model with group structure, called *Néron model*.

The main goal of this section is to give a criterion to show whether a given smooth projective curve C over a field K admits a stable model or not. Before that, we need to introduce the notion of the regular fibered surface, as we will see that every smooth projective curve over a field K admits a model, which is a regular fibered surface.

2.2. **Regular fibered surface.**

2.2.1. *Fibered surface and desingularization.*

Definition 2.2.1. Let S be a Dedekind scheme with generic point η.

(1) An integral, projective, flat S-scheme $\pi : X \to S$ of dimension two is called a *fibered surface* over S.
(2) A fiber X_s with $s \in S$ closed in called a *closed fiber*.
(3) The fiber X_η is called the *generic fiber*.

Remark 2.2.1. Let $X \to S$ be a fibered surface.

(1) If $\dim S = 0$, then X is an integral, projective algebraic surface over a field.
(2) If $\dim S = 1$, we can say that X is a "relative curve" over S.

Proposition 2.2.1. Let C be a smooth projective curve over a field K. Then C admits a model $\mathcal{C} \to S$ with affine S, which is a fibered surface with generic fiber isomorphic to C.

Proof. Firstly, let $\mathcal{C}_0 \to S$ be as in Example 2.1.2, where S is affine. Let $\mathcal{C}$ be the Zariski closure of C in $\mathcal{C}_0$, endowed with the reduced closed subscheme structure. Then $\mathcal{C} \to S$ is a fibered surface with generic fiber isomorphic to C. $\square$

Theorem 2.2.1 (Abhyankar,Hironaka,Lipman)**.** If X is an excellent, reduced, Noetherian scheme of dimension two, then there exists $\pi: \widetilde{X} \to X$ projective, birational morphism such that $\widetilde{X}$ is regular, and $\pi^{-1}(X_{\mathrm{reg}}) \cong X_{\mathrm{reg}}$.

Corollary 2.2.1. Let C be a smooth projective curve over a field K. Then C admits a model, which is a regular fibered surface.

Proof. Let $\mathcal{C} \to S$ be a model of C with fibered surface structure as proved in Proposition 2.2.1. Then by Theorem 2.2.1, there exists a regular fibered surface $\widetilde{\mathcal{C}} \to S$ which isomorphic to $\mathcal{C}$ on regular locus. In particular, we have $\widetilde{\mathcal{C}}_K \cong C$ since $C \subset \mathcal{C}_{\mathrm{reg}}$. This shows $\widetilde{\mathcal{C}}$ is a model of C over S, which is a regular fibered surface. □

2.2.2. *Backgrounds on intersection theory.* Let X be a regular, Noetherian, connected scheme of dimension two. Let D, E be two effective divisors on X with no common irreducible component. Let $x \in X$ be a closed point. Since $\operatorname{supp} D \cap \operatorname{supp} E = \{x\}$ or in a neighborhood of x, we have

$$\sqrt{\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x} = \mathfrak{m}_x \mathcal{O}_{X,x}.$$

Hence $\mathcal{O}_{X,x}/(\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x)$ is an Artinian ring, and consequently of finite length.

Definition 2.2.2. Let D, E be two effective divisors on X with no common irreducible component.

(1) Let $x \in X$ be a closed point. The integer

$$i_x(D,E) = \operatorname{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/(\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x).$$

is called *intersection number* of D and E at x.

(2) The *intersection cycle* $D.E$ is a 0-cycle

$$D.E = \sum_{x \in X} i_x(D,E)[x].$$

Remark 2.2.2. The intersection number is a non-negative integer, and $i_x(D,E) = 0$ if and only if $x \notin \operatorname{supp} D \cap \operatorname{supp} E$.

But how to define the intersection number when two divisors have common irreducible components, such as the self-intersection?

Lemma 2.2.1 (moving lemma)**.** Let X be a normal, Noetherian, connected, separated scheme. Let D, E be two Weil divisors on X. Then there exists an $f \in K(X)$ such that $\operatorname{div}(f) + D$ and E have no common component.

2.2.3. *Intersection theory on regular fibered surface.* Let $X \to S$ be a regular fibered surface over a Dedekind scheme S. If $\dim S = 1$, then X is called an *arithmetic surface*. In general it is impossible to define the intersection of two arbitrary divisors on X, but this obstacle can be circumvented when one of the divisors is vertical.

Definition 2.2.3. Let $x \in X$ be a closed point. A Cartier divisor E of X is called a *vertical divisor* over s, if $\operatorname{supp} E \subseteq X_s$. The set $\operatorname{Div}_s(X)$ consists of all Cartier divisors of X which are vertical over s.

Remark 2.2.3. If $\dim S = 1$, then $\mathrm{Div}_s(X)$ has the irreducible components of X_s as a basis.

Theorem 2.2.2. Let $X \to S$ be a regular fibered surface. Let $s \in S$ be a closed point. Then there exists a unique bilinear map

$$i_s : \mathrm{Div}(X) \times \mathrm{Div}_s(X) \to \mathbb{Z}$$

which verifies the following properties:

(1) If $D \in \mathrm{Div}(X)$ and $E \in \mathrm{Div}_s(X)$ have no common component, then

$$i_s(D,E) = \sum_x i_x(D,E)[k(x):k(s)],$$

where x runs through the closed points of X.

(2) The restriction of i_s to $\mathrm{Div}_s(X) \times \mathrm{Div}_s(X)$ is symmetric.

(3) $i_s(D,E) = i_s(D',E)$ if D is linearly equivalent to D'.

(4) If $0 < E \le X_s$, then

$$i_s(D,E) = \deg_{k(s)} \mathcal{O}_X(D)|_E.$$

Proposition 2.2.2. Let $X \to S$ be an arithmetic surface and $s \in S$ be a closed point. Then

(1) For any $E \in \mathrm{Div}_s(X)$, we have $E.X_s = 0$.

(2) Let $\Gamma_1, \dots, \Gamma_r$ be the irreducible components of X_s of respective multiplicities $d_1, \dots, d_r$. Then for any $i \le r$, we have

$$\Gamma_i^2 = -\frac{1}{d_i} \sum_{j \ne i} d_j \Gamma_i.\Gamma_j.$$

Theorem 2.2.3. Let $X \to S$ be an arithmetic surface with geometrically connected generic fiber. Then

$$\mathrm{Div}_s(X)_{\mathbb{R}} \times \mathrm{Div}_s(X)_{\mathbb{R}} \to \mathbb{R}$$

is negative semi-definite, and $E^2 = 0$ if and only if $E \in \mathbb{R} X_s$.

Proof. Suppose $X_s = \sum_i d_i \Gamma_i$, where Γ_i's are irreducible components and d_i is the multiplicity of Γ_i. For simplicity we denote $a_{ij} = \Gamma_i.\Gamma_j \ge 0$, $a_{ij} \ge 0$ when $i \ne j$ and $b_{ij} = a_{ij} d_i d_j$. Since $\Gamma_i.X_s = \sum_i d_j a_{ij} = 0$, it gives $\sum_i b_{ij} = 0$.

For any $V \in \mathrm{Div}_s(X)_{\mathbb{R}}$, we write it as $V = \sum_i x_i \Gamma_i$. Denote $y_i = x_i / d_i$. Then

$$\begin{aligned} V.V &= \left(\sum_i d_i y_i \Gamma_i\right)\left(\sum_j d_j y_j \Gamma_j\right) \\ &= \sum_{i,j} y_i y_j d_i d_j a_{ij} \\ &= -\sum_{i>j} b_{ij} (y_i - y_j)^2 \\ &\le 0 \end{aligned}$$

The equality holds if and only if $b_{ij}(y_i - y_j) = 0$ for all $i < j$. Note that $y_i = y_j$ if $\Gamma_i.\Gamma_j \ne 0$, that is, $\Gamma_i \cap \Gamma_j \ne \varnothing$. Since the generic fiber of X is geometrically connected, then Zariski's theorem implies X_s is connected. □

Proposition 2.2.3. Let $X \to S$ be an arithmetic surface and $s \in S$ be a closed point. Let $\Gamma_1, \dots, \Gamma_r$ be the irreducible components of X_s with respective multiplicities $d_1, \dots, d_r$. Then

$$2p_a(X_\eta) - 2 = \sum_{i=1}^{r} d_i \omega_{X/S}.\Gamma_i.$$

Proof. By adjunction formula we have

$$\omega_{X_s/k(s)} \cong \omega_{X/S}|_{X_s} \otimes \mathcal{O}_X(X_s)|_{X_s}.$$

Then intersect with X_s it gives

$$2p_a(X_s) - 2 = \omega_{X/S}.X_s = \sum_{i=1}^{r} d_i \omega_{X/S}.\Gamma_i.$$

Since $X \to S$ is flat, we have $2p_a(X_s) - 2 = 2g(X_\eta) - 2$. This completes the proof. □

2.2.4. *Minimal regular model.* Let $X \to S$ be a regular fibered surface.

Definition 2.2.4. An irreducible divisor E of X is called an *exceptional divisor* if there exists a regular fibered surface $Y \to S$ and a morphism $f : X \to Y$ of S-schemes such that $f(E)$ is a point and $f : X \setminus E \to Y \setminus f(E)$ is an isomorphism.

Theorem 2.2.4 (Castelnuovo's criterion)**.** Let $X \to S$ be a regular fibered surface. Let $E \subset X_s$ be an irreducible divisor. Then E is an exceptional divisor if and only if $E \cong \mathbb{P}^1_{k'}$ and $E^2 = -[k' : k(s)]$, where $k' = H^0(E, \mathcal{O}_E)$.

Definition 2.2.5. Let $X \to S$ be a regular fibered surface. Then

(1) It is called *relatively minimal* if for all closed point $s \in S$, the fiber X_s has no exceptional curve.
(2) It is called *minimal* every birational map of regular fibered S-surfaces $Y \dashrightarrow X$ is a birational morphism.

Theorem 2.2.5 ([Liu02, Theorem 9.3.21])**.** Let $X \to S$ be an arithmetic surface with generic fiber of genus $g(C_\eta) \geq 1$. Then X admits a unique minimal model over S, up to unique isomorphism.

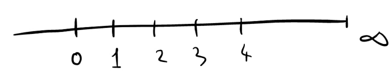
Corollary 2.2.2 ([Liu02, Corollary 9.3.24])**.** Let $X \to S$ be a relatively minimal arithmetic surface with generic fiber $g(X_\eta) \geq 1$. Then X is minimal.

Proposition 2.2.4. Let C be a smooth projective curve over a field K and S be a Dedekind scheme with function field K. Then C has a stable model over S if and only if the minimal model $\mathcal{C}^{\min}$ over S is semi-stable.

Example 2.2.1. *Let $S = \operatorname{Spec}\mathbb{C}[t]$ and $\mathcal{C} \to S$ is defined by the compactification of the affine plane curve $y^2 = tx(x-1)(x-2)(x-3)(x-4)$. By Jacobi criterion $\mathcal{C}$ is not regular, with singular locus*

$$\mathcal{C}_{\text{Sing}} = \{[0:0:x] \mid x = 0,1,2,3,4,\infty\} \subset \mathbb{P}^1_{\mathbb{C}},$$

that is,



By blow-up these six singular points we have

These components are isomorphic $\mathbb{P}^1_{\mathbb{C}}$ with self-intersection -2, and thus this gives the minimal model $\mathcal{C}^{\min}$ over S, but it is not semi-stable.

Remark 2.2.4. Let t_1 be a variable such that $t_1^2 = t$ and consider $\mathbb{C}[t] \subset \mathbb{C}[t_1]$. Then $\mathcal{C} \to S' = \operatorname{Spec}\mathbb{C}[t_1]$ defined by $y^2 = tx(x-1)(x-2)(x-3)(x-4)$ is a smooth model over S', since we have

$$\left(\frac{y}{t_1}\right)^2 = x(x-1)(x-2)(x-3)(x-4).$$

This is a simple example of Deligne-Mumford theorem.

3. DELIGNE-MUMFORD THEOREM

3.1. **Simple normal crossing regular fibered surface.** Let C be a smooth projective curve over an algebraically closed field K with $\mathrm{char}(K) = 0$ and S be a Dedekind scheme with function field K.

Theorem 3.1.1 (Deligne-Mumford)**.** There exists a finite map $S' \to S$, where S' is a Dedekind scheme and $K' = K(S') \to K$ is separable, such that C/K' has a stable model.

Definition 3.1.1. A regular model $\mathcal{C} \to S$ is called *simple normal crossing (SNC)*, if for every closed point $s \in S$, every irreducible component Γ_i of $\mathcal{C}_s$ is smooth, and $p \in \Gamma_i \cap \Gamma_j$ is a node.

Proposition 3.1.1. There exists a SNC regular model.

Proof. We start with a regular model and consider resolution of singularity of singular fiber. ☐

Proposition 3.1.2. Let $\mathcal{C} \to S$ be a SNC regular model and $p \in \mathcal{C}_s$ be a point in a closed fiber.

(1) If p is not an intersection point, then we have $\mathcal{C} \hookrightarrow Z \to S$, where $\mathcal{Z} \to S$ is smooth and $\dim Z_s = 2$. Moreover, locally at p we have
$$\mathcal{O}_{\mathcal{C},p} = \frac{\mathcal{O}_{\mathcal{Z},p}}{(u^d - ta)},$$
where t is uniformizer of $\mathcal{O}_K$ and $a \in \mathcal{O}^*_{Z,p}$.

(2) If $p \in \Gamma_1 \cap \Gamma_2$, then there exists $\mathcal{Z}$ as above, and locally at p we have
$$\mathcal{O}_{\mathcal{C},p} = \frac{\mathcal{O}_{\mathcal{Z},p}}{(u^{d_1} v^{d_2} - ta)},$$
where t is uniformizer of $\mathcal{O}_K$ and $a \in \mathcal{O}^*_{Z,p}$.

Theorem 3.1.2 ([Liu02, Proposition 10.4.6])**.** Let $\mathcal{C} \to S$ be a SNC regular model and $\mathcal{C}_s = \sum_i d_i \Gamma_i$ be the irreducible components. Let $e = \mathrm{lcm}(d_i)$ and $\mathcal{O}_L = \mathcal{O}_K[t^{\frac{1}{e}}]$. Then the normalization of $\mathcal{C} \times_{\mathcal{O}_K} \mathcal{O}_L$ is a semi-stable model over $\mathcal{O}_L$.

Lemma 3.1.1. If C has stable reduction over an étale extension L/K. Then C has a stable reduction over K.

Remark 3.1.1. If C has a stable model $\mathcal{C} \to S$. Then for any finite $S' \to S$, C/K' has a stable model over S', which is exactly $\mathcal{C} \times_S S' \to S'$.

3.2. **Néron model of abelian variety.** Let S be a Dedekind scheme with function field K and A be an abelian variety over K.

Definition 3.2.1. A *Néron model* $\mathcal{A}$ of A is a smooth, separated model of A such that for any smooth scheme $\mathfrak{X} \to S$, there exists an one to one correspondence
$$\mathrm{Mor}_S(\mathfrak{X}, \mathcal{A}) \Longleftrightarrow \mathrm{Mor}_K(\mathfrak{X}_K, A).$$

Remark 3.2.1. Let $\mathcal{A} \to S$ be a Néron model of A over S. Then for any smooth scheme $\mathfrak{X} \to S$ and any $f_K : \mathfrak{X}_K \to A$, there exists a unique extension $f : \mathfrak{X} \to \mathcal{A}$. In particular,

$$\mathcal{A}(S) \Longleftrightarrow A(K)$$

For any closed point $s \in S$, there is a natural morphism $\mathcal{A}(S) \to \mathcal{A}(k(s))$, we can composite it with the bijection between $\mathcal{A}(S)$ and $A(K)$ to obtain the so-called *reduction map*. For example. Let A be an abelian variety over $\mathbb{Q}$. Then $S = \operatorname{Spec} \mathbb{Z}$ and

$$A(\mathbb{Q}) \to \mathcal{A}_{\mathbb{F}_p}(\mathbb{F}_p)$$

is the reduction map.

Proposition 3.2.1.

(1) Néron model is unique up to a unique isomorphism.
(2) Néron model is a group scheme, since generic fiber is an abelian variety, so multiplication, addition maps can uniquely extend to the Néron model $\mathcal{A}$.

$$\begin{tikzcd}
A \times_k A \arrow[r] \arrow[d] & A \arrow[d] \\
\mathcal{A} \times_S \mathcal{A} \arrow[r] & \mathcal{A}
\end{tikzcd}$$

Theorem 3.2.1 (Néron, Raynaud)**.** The Néron model exists.

Example 3.2.1. *Let A be an elliptic curve over K. Then minimal projective regular model $\mathcal{A}^{\min}$ over S. Then Néron model is $(\mathcal{A}^{\min})_{sm}$.*

Let $\mathcal{A} \to S$ be a commutative group scheme over S. For any closed point $s \in S$, there exists the following exact sequence

$$0 \to (\mathcal{A}_s)^0 \to \mathcal{A}_s \to \phi_s \to 0,$$

where $(\mathcal{A}_s)^0$ is the connected component of identity and ϕ_s is étale finite algebraic group. By Chevalley's theorem we have

$$0 \to L \to \mathcal{A}_s^0 \to B \to 0,$$

where L is a linear group and B is an abelian variety over $k(s)$. If $k(s)$ is a perfect field, then $L \cong T \times U$, where T is a torus and U is a nilpotent group.

Definition 3.2.2. Let $\mathcal{A} \to S$ be the Néron model.

(1) We say that $\mathcal{A}$ has a good reduction at s if $\mathcal{A}_s$ is projective.
(2) We say that $\mathcal{A}$ has a semi-abelian at s if the nilpotent part of $\mathcal{A}_s$ is zero.

Theorem 3.2.2 (Grothendieck)**.** There exists a finite $S' \to S$, where S' is Dedekind scheme with function field K', such that $A_{K'}$ has a semi-abelian reduction.

Theorem 3.2.3 (Deligne-Mumford)**.** Let C be a smooth projective curve over a field k with genus ≥ 1. Then C has stable reduction if and only if $\operatorname{Jac}(C)$ has semi-abelian reduction.

Idea of the proof. Let $\mathcal{C}$ be the minimal regular model of C. It gives a functor $\mathrm{Pic}^0_{\mathcal{C}/S}$, which is represented by a commutative smooth group schemes if $C(K) \neq \varnothing$ (Raynaud), and it is also isomorphic to $\mathcal{A}^0$, where $\mathcal{A}$ is Néron model of $\mathrm{Jac}(C)$. This shows $\mathcal{A}^0_s = \mathrm{Pic}^0_{\mathcal{C}_s/k(s)}$. Thus it suffices to consider the picard group of nodal curves. $\square$

Appendix A. Kähler differential

A.1. **Local picture.** Let k be a field and A be a k-algebra.

Definition A.1.1. Let M be an A-module. A *derivation* is a map $\mathrm{d}\colon A \to M$ such that

(1) d is k-linear;
(2) d satisfies Leibniz rule, that is, for any $a, b \in A$,

$$\mathrm{d}(a_1 a_2) = a_1 \mathrm{d}a_2 + a_2 \mathrm{d}a_1.$$

The set of all derivations is a A-module, which is denoted by $\mathrm{Der}_k(A, M)$.

Example A.1.1. *Let $A = k[x_1, \ldots, x_n]$. To determine a derivation $\mathrm{d}\colon A \to M$ is equivalent to determine $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\} \subset M$. Thus*

$$\mathrm{Der}_k(k[x_1, \ldots, x_n], M) = M^{\oplus n}$$

as A-modules.

Example A.1.2. *Let A, B be k-algebras and $\varphi_0\colon A \to B$ be a k-algebra homomorphism. A k-algebra homomorphism*

$$\varphi\colon A \to B \otimes k[\epsilon]/(\epsilon^2)$$

such that $\varphi(a) = \varphi_0(a) + \epsilon\varphi_1(a)$ is called a deformation *of φ_0.*

In order to determine all possible deformations of φ_0, it suffices to determine all possible φ_1. For any $a, b \in A$, the condition $\varphi(ab) = \varphi(a)\varphi(b)$ is equivalent to $\varphi_1 \in \mathrm{Der}_k(A, B)$. In other words, the set of deformations of $\varphi_0\colon A \to B$ is in one to one correspondence with the set of derivations $\mathrm{Der}_k(A, B)$.

Definition A.1.2. The *Kähler differential* $\Omega^1_{A/k}$ is an A-module together with a k-derivation d_A such that

$$\mathrm{Hom}_A(\Omega^1_{A/k}, M) \xleftrightarrow{1-1} \mathrm{Der}_k(A, M)$$
$$\varphi \mapsto \varphi \circ \mathrm{d}_A.$$

Proposition A.1.1 ([Liu02, Proposition 1.8])**.** Let A be a k-algebra.

(1) For any field extension $k \subseteq k'$, let us set $A' = A \otimes_k k'$. Then there exists a canonical isomorphism

$$\Omega_{A'/k'} = \Omega_{A/k} \otimes_k k'.$$

(2) Let S be a multiplicative subset of A. Then

$$S^{-1}\Omega^1_{A/k} \cong \Omega^1_{S^{-1}A/k}.$$

(3) Let $A \to B$ be a surjective morphism of k-algebras with kernel I. Then there is an exact sequence of S-modules

$$I/I^2 \to \Omega^1_{A/k} \otimes_k B \to \Omega^1_{B/k} \to 0,$$

where $[f] \in I/I^2$ maps to $1 \otimes \mathrm{d}f \in \Omega^1_{A/k} \otimes_k B$.

A.2. **The sheaf of Kähler differential.**

Definition A.2.1. Let X be an algebraic variety over a field k. The *sheaf of Kähler differential* Ω^1_X is the sheaf of $\mathcal{O}_X$-modules defined by

$$\Omega^1_X(U) := \Omega^1_{\mathcal{O}_X(U)/k}$$

on affine open subsets U.

Proposition A.2.1 ([Liu02, Proposition 2.2])**.** Let X be an algebraic variety over a field k of (pure) dimension d. Then X is smooth if and only if $\Omega^1_{X/k}$ is locally free of rank d.

Definition A.2.2. Let X be a smooth algebraic variety over a field k. The *dualizing sheaf* $\omega_{X/k}$ is defined as

$$\omega_{X/k} := \det \Omega^1_{X/k}.$$

Theorem A.2.1 (Serre duality)**.** Let X be a smooth algebraic variety over a field k of dimension d. For any coherent sheaf $\mathcal{F}$ on X, there exists a canonical non-degenerate pairing

$$H^d(X,\mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_{X/k}) \to H^d(X,\omega_{X/k}) \cong k.$$

A.3. **Dualizing sheaf on locally complete intersection.** In this section, we will introduce how to defined the dualizing sheaf $\omega_{X/k}$ on an algebraic variety of locally complete intersection.

Definition A.3.1. Let A be a ring and $a_1,\dots,a_n$ be a sequence of elements of A. It is a *regular sequence*, if a_1 is not a zero divisor, and a_i is not a zero divisor in $A/(a_1,\dots,a_{i-1})$ for any $i \geq 2$.

Lemma A.3.1. Let A be a ring and I be an ideal generated by a regular sequence $a_1,\dots,a_n \in \mathfrak{m}$. Then I/I^2 is a free A/I-module of rank n.

Proof. If there exist $a_1,\dots,a_n \in A$ such that $x_1a_1 + \cdots + x_na_n \in I^2$, then we need to prove $x_1,\dots,x_n \in I$. Firstly we prove that if $\sum_{i=1}^n x_ia_i = 0$, then $x_i \in I$, by induction on n.

If $n = 1$, we have $x_1a_1 = 0$ implies $x_1 = 0$ since a_1 is not a zero divisor. For $n \geq 2$, we have

$$x_na_n = -x_{n-1}a_{n-1} - \cdots - x_1a_1.$$

By passing to the quotient ring $A/(a_1,\dots,a_{n-1})$, it gives $\overline{x}_n\overline{a}_n = 0$, and thus $x_n \in (a_1,\dots,a_{n-1})$ since $\overline{a}_n$ is not a zero divisor in $A/(a_1,\dots,a_{n-1})$. Suppose $x_n = \sum_{i=1}^{n-1} a_iy_i$ with $y_i \in A$. It follows that

$$\sum_{i=1}^{n-1} a_i(x_i + a_ny_i) = 0,$$

and by induction hypothesis it gives $x_i + a_ny_i \in (a_1,\dots,a_{n-1})$, hence $x_i \in I$, and we have already proved $a_n \in I$.

Now we back to the proof the freeness of I/I^2 as A/I-module. Since $x_1a_1 + \cdots + x_na_n \in I^2$, there exist $z_1, \ldots, z_n \in I$ such that

$$\sum_{i=1}^{n} a_i(x_i - z_i) = 0,$$

since $I^2 = \sum_{i=1}^{n} a_i I$. From the above, we have $x_i - z_i \in I$, and thus $x_i \in I$. □

Definition A.3.2. Let X be an algebraic variety over a field k and $Z \subseteq X$ be a closed subvariety. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then $Z \hookrightarrow X$ is a *regular immersion* if for all $x \in X$, the stalk $\mathcal{I}_{Z,x}$ is generated by a regular sequence.

Definition A.3.3. Let X be an algebraic variety over a field k and $i\colon Z \hookrightarrow X$ be a closed subvariety. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then $\mathcal{C}_{Z/X} := i^*\left(\mathcal{I}_Z/\mathcal{I}_Z^2\right)$ is called *conormal bundle* of Z in X.

Corollary A.3.1. Let X be an algebraic variety over a field k and $i\colon Z \hookrightarrow X$ be a regular immersion. Then the conormal bundle $\mathcal{C}_{Z/X}$ is locally free.

Definition A.3.4. Let X be an algebraic variety over a field k.

(1) X is called *locally complete intersection at* $x \in X$, if there exists a neighborhood U of x and a regular immersion $i\colon U \to Z$, where Z is a smooth algebraic variety over a field k.
(2) X is called *locally complete intersection*, if X is locally complete intersection at every point $x \in X$.

Example A.3.1. *Any smooth algebraic variety is a locally complete intersection.*

Lemma A.3.2 ([Liu02, Lemma 6.3.21])**.** Let X be an algebraic variety over a field k which is a locally complete intersection. Then any immersion $X \hookrightarrow Y$ is a regular immersion, where Y is an algebraic variety over a field k.

Lemma A.3.3. Let X be a reduced variety over a field k which is a locally complete intersection. Suppose $X \hookrightarrow Y$ is a regular immersion, where Y is a smooth variety over a field k. Then

$$0 \to \mathcal{C}_{X/Y} \to \Omega^1_{Y/k}\big|_X \to \Omega^1_{X/k} \to 0$$

is exact.

Proof. If X is smooth, then

$$0 \to \mathcal{C}_{X/Y} \to \Omega^1_{Y/k}\big|_X \to \Omega^1_{X/k} \to 0$$

is exact and split. If X is reduced, then there exists an open dense subset $U \subset X$ such that U is smooth. Thus $\mathcal{C}_{X/Y} \to \Omega^1_{Y/k}\big|_X$ is injective on U, that is,

$$\ker\left\{\mathcal{C}_{X/Y} \to \Omega^1_{Y/k}\big|_X\right\}$$

supports in $X \setminus U$. Since X is a locally complete intersection, we have conormal bundle $\mathcal{C}_{X/Y}$ is locally free, which implies the kernel must be zero. □

Definition A.3.5. Let X be an algebraic variety over a field k which is a locally complete intersection. Suppose $X \hookrightarrow Y$ is a regular immersion, where Y is a smooth variety over a field k. The dualizing sheaf of X is defined by

$$\omega_{X/k} := \omega_{Y/k}|_X \otimes \det(C_{Y/X})^\vee$$

Remark A.3.1. Note that the dualizing sheaf of X is independent of the choice of regular immersion $X \hookrightarrow Y$ ([Liu02, Lemma 6.4.5]).

Lemma A.3.4 ([Liu02, Corollary 6.4.14])**.** Let $X = V(F_1, \dots, F_r) \subset \mathbb{A}^n_k = \operatorname{Spec} k[T_1, \dots, T_n]$ and $(F_1, \dots, F_r)$ is a regular sequence. Suppose X is integral and

$$\Delta = \det\left(\frac{\partial F_i}{\partial T_j}\right)_{1 \le i,j \le r}$$

is non-zero in $K(X)$. Then

$$\omega_{X/k} = \frac{1}{\Delta}(\mathrm{d}T_{r+1} \wedge \cdots \wedge \mathrm{d}T_n)\mathcal{O}_X.$$

Example A.3.2. *Let C be the algebraic curve over a field k defined by the equation $y^2 + xy - x^3 = 0$. It is clear that $\Delta = 2y + x$ is non-zero in $K(C)$. Then $\omega_{C/k} = \frac{\mathrm{d}x}{2y+x}\mathcal{O}_C$. The curve C has a unique singular point $p = (0,0)$ of multiplicity one and the normalization of C is given by*

$$\begin{aligned} \pi : \widetilde{C} &\to C \\ t &\mapsto \left(t^2 + t, t^2(t+1)\right). \end{aligned}$$

This shows the singular point p is a node. Note that

$$\omega_0 = \frac{\mathrm{d}x}{2y+x} = \frac{\mathrm{d}t}{t(t+1)}.$$

This shows ω_0 is a rational differential form on $\mathbb{A}^1$, with simple pole at $t = 0$ and $t = -1$ such that

$$\operatorname{Res}_{t=0}\omega_0 + \operatorname{Res}_{t=-1}\omega_0 = 0.$$

For any $a \in \mathcal{O}_C$, we have $\omega_C = a\omega_0$ is a rational form on $\widetilde{C}$, and

$$\begin{aligned} \operatorname{Res}_{t=0}\omega_C + \operatorname{Res}_{t=-1}\omega_C &= a(t=0) - a(t=-1) \\ &= 0. \end{aligned}$$

This shows ω_C is a rational differential form on $\widetilde{C}$ with simple pole at $\pi^{-1}(p)$ such that $\operatorname{Res}_{p_1}\eta + \operatorname{Res}_{p_2}\eta = 0$. Conversely, for any rational differential form $\eta = b\mathrm{d}t$ on $\widetilde{C}$ with simple pole at $\pi^{-1}(p) = \{p_1, p_2\}$ such that $\operatorname{Res}_{p_1}\eta + \operatorname{Res}_{p_2}\eta = 0$, we have $b(p_1) = b(p_2)$.

References

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