Moduli problems of vector bundles

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Contents

PREFACE

It's a learning note for moduli problems of vector bundles. The main reference for moduli space and geometric invariant theory is [Hos16], and the main reference for algebraic stack is [Hei10].

Assumptions.

(1) k always denotes an algebraically closed field;
(2) By a scheme we always mean a finite type scheme over k;
(3) By a variety we always mean a reduced seperated scheme over k;

1. Moduli problems

1.1. Functors of points.

Definition 1.1.1. The *functor of points* of a scheme X is a contravariant functor $h_X := \mathrm{Hom}(-,X)\colon \mathrm{Sch} \to \mathrm{Set}$, and a morphism of schemes $f\colon X \to Y$ induces a natural transformation of funtors $h_f\colon h_X \to h_Y$, given by

$$\begin{aligned} h_f(Z)\colon h_X(Z) &\to h_Y(Z) \\ g &\mapsto f \circ g, \end{aligned}$$

where Z is a scheme.

Definition 1.1.2. The contravariant functors from Sch to Set are called *presheaves* on Sch. The category of presheaves on Sch is denoted by $\mathrm{Psh}(\mathrm{Sch}) = \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set})$.

Example 1.1.1. *For a scheme X, $h_X(\mathrm{Spec}\, k) = \mathrm{Hom}(\mathrm{Spec}\, k, X)$ is the set of k-valued points of X.*

Lemma 1.1.1 (Yoneda lemma)**.** Let $\mathscr{C}$ be any category. Then for any $C \in \mathscr{C}$ and any presheaf $\mathcal{F} \in \mathrm{Psh}(\mathscr{C})$, there is a bijection

$$\{\text{natural transformsations } \eta\colon h_C \to \mathcal{F}\} \longleftrightarrow \mathcal{F}(C),$$

which is given by $\eta \mapsto \eta_C(\mathrm{id}_{\mathscr{C}})$.

Proof. To see the surjectivity: For an object $s \in \mathcal{F}(C)$, we define $\eta\colon h_C \to \mathcal{F}$ defined as follows: For $C' \in \mathscr{C}$, consider

$$\begin{aligned} \eta_{C'}\colon h_C(C') &\to \mathcal{F}(C') \\ f &\mapsto \mathcal{F}(f)(s). \end{aligned}$$

(1) It's well-defined: Since $\mathcal{F}$ is a contravariant functor, then for $f\colon C' \to C$, we have $\mathcal{F}(f)\colon \mathcal{F}(C) \to \mathcal{F}(C')$, and thus $\mathcal{F}(f)(s) \in \mathcal{F}(C')$.
(2) It's a natural transformation: Since if we take $g\colon C'' \to C'$, and consider the following diagram

$$\begin{array}{ccc} h_C(C') & \xrightarrow{\eta_{C'}} & \mathcal{F}(C') \\ \big\downarrow{\scriptstyle h_C(g)} & & \big\downarrow{\scriptstyle \mathcal{F}(g)} \\ h_C(C'') & \xrightarrow{\eta_{C''}} & \mathcal{F}(C''). \end{array}$$

For arbitrary $f\colon C' \to C \in h_C(C')$, note that

$$\begin{aligned} \eta_{C''} \circ h_C(g) &= \eta_{C''}(f \circ g) \\ &= \mathcal{F}(f \circ g)(s) \\ &= \mathcal{F}(g) \circ \mathcal{F}(f)(s) \\ &= \mathcal{F}(g) \circ \eta_{C'}(f). \end{aligned}$$

Thus above diagram commutes, that is, η is a natural transformation.

By construction, we have

$$\eta_C(\mathrm{id}_C) = \mathcal{F}(\mathrm{id}_C)(s) = s.$$

This proves the surjectivity.

To see the injectivity: Suppose we have two natural transformation $\eta, \eta' : h_C \to \mathcal{F}$ such that $\eta_C(\mathrm{id}_C) = \eta'_C(\mathrm{id}_C)$. Then if we want to show $\eta = \eta'$, it suffices to show for arbitrary $C' \in \mathscr{C}$, we have $\eta_{C'} = \eta'_{C'}$. Let $g : C' \to C$. Then we have the following commutative diagram

$$\begin{array}{ccc} h_C(C) & \xrightarrow{\eta_C} & \mathcal{F}(C) \\ \downarrow{\scriptstyle h_C(g)} & & \downarrow{\scriptstyle \mathcal{F}(g)} \\ h_C(C') & \xrightarrow{\eta_{C'}} & \mathcal{F}(C'). \end{array}$$

It follows that

$$\mathcal{F}(g) \circ \eta_C(\mathrm{id}_C) = \eta_{C'} \circ h_C(g)(\mathrm{id}_C) = \eta_{C'}(g),$$

and by the same argument one has $\mathcal{F}(g) \circ \eta'_C(\mathrm{id}_C) = \eta'_{C'}(g)$. Hence

$$\eta_{C'}(g) = \mathcal{F}(g) \circ \eta_C(\mathrm{id}_C) = \mathcal{F}(g) \circ \eta'_C(\mathrm{id}_C) = \eta'_{C'}(g).$$

This completes the proof. □

Corollary 1.1.1. The functor $h : \mathscr{C} \to \mathrm{Psh}(\mathscr{C})$ is fully faithful.

Proof. A functor is called fully faithful if for every $C, C' \in \mathscr{C}$, there is the following bijection

$$\mathrm{Hom}_{\mathscr{C}}\left(C, C'\right) \leftrightarrow \mathrm{Hom}_{\mathrm{Psh}(\mathscr{C})}(h_C, h_{C'}).$$

Then take $\mathcal{F} = h_{C'}$ in Yoneda lemma to conclude. □

Definition 1.1.3. A presheaf $\mathcal{F} \in \mathrm{Psh}(\mathscr{C})$ is called *reprensentable* if there exists an object $C \in \mathscr{C}$ and a natural isomorphism $F \cong h_C$.

So it's natural to ask if every presheaf F is reprensentable by a scheme X? The answer is negative, as we will see. However, we are quite interested in answering this question for special functors, known as moduli functor.

1.2. **Moduli problems.** A moduli problem is a classification problem: we have a collection of objects and we want to classify them up to some equivalence. In fact, we want more than this: we want a moduli space encodes how these objects vary continously in families.

Definition 1.2.1. A *naive moduli problem* (in algebraic geometry) is a collection $\mathcal{A}$ of objects (in algebraic geometry) and an equivalence relation $\sim$ on $\mathcal{A}$.

Example 1.2.1.

1. Let $\mathcal{A}$ be the set of k-dimensional linear subspaces of an n-dimensional vector space and $\sim$ be equality.
2. Let $\mathcal{A}$ be the collection of vector bundles on a fixed scheme X and $\sim$ be the relation given by isomorphism of vector bundles.

Our aim is to find a scheme M whose k-points are in bijection with equivalence classes $\mathcal{A}/_\sim$. Furthermore, we want M to also encode how these objects vary continously in "families".

Definition 1.2.2. Let $(\mathcal{A}, \sim)$ be a naive moduli problem. Then a *moduli problem* is given by

1. sets $\mathcal{A}_S$ of families over S and an equivalence relation $\sim_S$ on $\mathcal{A}_S$ for all schemes S.
2. pullback maps $f^*\colon \mathcal{A}_S \to \mathcal{A}_T$, for every morphism of schemes $f\colon T \to S$, such that
   (1) $(\mathcal{A}_{\operatorname{Spec} k}, \sim_{\operatorname{Spec} k}) = (\mathcal{A}, \sim)$;
   (2) For the identity $\operatorname{id}\colon S \to S$ and any family $\mathcal{F}$ over S, we have $\operatorname{id}^* \mathcal{F} = \mathcal{F}$;
   (3) For a morphism $f\colon T \to S$ and equivalent families $\mathcal{F} \sim_S \mathcal{G}$, we have $f^*\mathcal{F} \sim_T f^*\mathcal{G}$.
   (4) For morphisms $f\colon T \to S$, $g : S \to R$, and a family $\mathcal{F}$ over R, we have an equivalence
   $$(g \circ f)^* \mathcal{F} \sim_T f^* g^* \mathcal{F}$$

Notation 1.2.1. For a family $\mathcal{F}$ over S and a point $s\colon \operatorname{Spec} k \to S$, $\mathcal{F}_s := s^*\mathcal{F}$ denotes the corresponding family over $\operatorname{Spec} k$.

Corollary 1.2.1. A moduli problem defines a functor $\mathcal{M} \in \operatorname{Psh}(\operatorname{Sch})$, given by

$$\mathcal{M}(S) := \{\text{families over } S\}/_{\sim_S}$$
$$\mathcal{M}(f\colon T \to S) := f^*\colon \mathcal{M}(S) \to \mathcal{M}(T)$$

Example 1.2.2. *Consider the naive moduli problem given by vector bundles on a fixed scheme X up to isomorphism. Then this can be extended in two different ways. The natural notion for a family over S is a locally free sheaf $\mathcal{F}$ over $X \times S$ which is flat over S, but there are two possible ways to define relations:*

$$\mathcal{F} \sim'_S \mathcal{G} \iff \mathcal{F} \cong \mathcal{G}$$
$$\mathcal{F} \sim_S \mathcal{G} \iff \mathcal{F} \cong \mathcal{G} \otimes \pi_S^* \mathcal{L}$$

where $\mathcal{L}$ is a line bundle $\mathcal{L} \to S$ and $\pi_S\colon X \times S \to S$.

1.3. **Fine moduli spaces.**

The ideal is when there is a scheme that reprensents our given moduli functor.

Definition 1.3.1. Let $\mathcal{M}\colon \operatorname{Sch} \to \operatorname{Set}$ be a moduli functor. Then a scheme M is a *fine moduli space* for $\mathcal{M}$ if it reprensents $\mathcal{M}$.

Remark 1.3.1. To be explicit, the scheme M is a fine moduli space for the moduli functor $\mathcal{M}$ if there is a natural isomorphism $\eta\colon \mathcal{M} \to h_M$. Thus for every scheme S, we have a bijection

$$\eta_S\colon \mathcal{M}(S) = \{\text{families over } S\}/_{\sim_S} \longleftrightarrow h_M(S) = \{\text{morphisms } S \to M\}$$

In particular, if $S = \operatorname{Spec} k$, then the k-points of M are in bijection with the set $\mathcal{A}/_\sim$. Moreover, if $T \to S$ is a morphism between schemes, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{\eta_S} & h_M(S) \\ \downarrow & & \downarrow \\ \mathcal{M}(T) & \xrightarrow[\eta_T]{} & h_M(T). \end{array}$$

Definition 1.3.2. Let M be a fine moduli space for $\mathcal{M}$. Then the family $\mathcal{U} \in \mathcal{M}(M)$, determined by $\mathcal{U} := \eta_M^{-1}(\mathrm{id}_M)$, is called the *universal family*.

Remark 1.3.2. For any $\mathcal{F} \in \mathcal{M}(S)$, that is, a family over a scheme S, it corresponds to a morphism $f: S \to M$. On the other hand, the family $f^*\mathcal{U}$ corresponds to the morphism $\mathrm{id}_M \circ f$.

$$\begin{array}{ccc} f^*\mathcal{U} \in \mathcal{M}(S) & \xrightarrow{\eta_S} & h_M(S) \ni \mathrm{id}_M \circ f \\ \downarrow & & \downarrow \\ \mathcal{U} \in \mathcal{M}(M) & \xrightarrow[\eta_T]{} & h_M(M) \ni \mathrm{id}_M. \end{array}$$

This shows families $f^*\mathcal{U}$ and $\mathcal{F}$ correspond to the same morphism, and thus

$$f^*\mathcal{U} \sim_S \mathcal{F}.$$

This shows any family is equivalent to a family obtained by pulling back the universal family, and that's why $\mathcal{U}$ is called the universal family.

Proposition 1.3.1. If a fine moduli space for moduli functor $\mathcal{M}$ exists, then it is unique up to a unique isomorphism.

Proof. Suppose $(M,\eta),(M',\eta')$ are two fine moduli spaces for the moduli functor $\mathcal{M}$. Then they are related by unique isomorphisms

$$\eta'_M \circ (\eta_M)^{-1}(\mathrm{id}_M) \colon M \to M',$$
$$\eta_{M'} \circ (\eta'_{M'})^{-1}(\mathrm{id}_{M'}) \colon M' \to M.$$

□

Example 1.3.1. *In this example let's show that the projective space $\mathbb{P}^n$ can be interpreted as a fine moduli space for the moduli problem of lines through the origin $V := \mathbb{A}^{n+1}$. Firstly, we need to clearify the definition of the moduli functor in this setting. A family of lines through the origin in V over a scheme S is a line bundle $\mathcal{L}$ over S which is a subbundle of the trivial vector bundle $V \times S$ over S, and two families are equivalent if and only if they are equal.*

There is a tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subseteq V \times \mathbb{P}^n$, and the dual line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ is generated by global sections $x_0,\dots,x_n$. Given any morphism $f: S \to \mathbb{P}^n$, the line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by the global sections $f^*(x_0),\dots,f^*(x_n)$. Hence there is a surjection $\mathcal{O}_S^{\oplus n+1} \twoheadrightarrow f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Dualize the above surjection we obtain an inclusion $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}^n}(1) \hookrightarrow \mathcal{O}_S^{\oplus n+1} = V \times S$. This provides a family of lines through the origin in V over S.

Conversely, let $\mathcal{L} \subseteq V \times S$ be a family of lines through the origin in V over S. Then dualize this inclusion this provides a surjection $q: V^* \times S \twoheadrightarrow \mathcal{L}^*$. The vector *bundle* $V^* \times S$ *is globally generated by the global sections* $\sigma_0, \ldots, \sigma_n$ *corresponding to the dual basis of the standard basis on* V. *In particular, it provides a unique morphism*

$$f : S \to \mathbb{P}^n$$
$$s \mapsto [q \circ \sigma_0(s), \ldots, q \circ \sigma_n(s)],$$

and $f^* \mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{L} \subseteq V \times S$ *by* [Har77, Theorem 7.1].

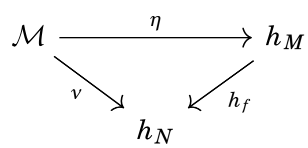
Hence, there is a bijective correspondence between morphisms $S \to \mathbb{P}^n$ *and families of lines through the origin in* V *over* S, *and this bijection has functoriality. In particular, the projective space* $\mathbb{P}^n$ *is a fine moduli space for this moduli problem and tautological line bundle is the universal family.*

Example 1.3.2. *The Grassmannian variety* $\mathrm{Gr}(d,n)$ *is a fine moduli space for the moduli problem of* d*-dimensional linear subspaces of a fixed vector space* $V = \mathbb{A}^n$.

1.4. **Coarse moduli spaces.**

Definition 1.4.1. A *coarse moduli space* for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation of funtors $\eta \colon \mathcal{M} \to h_M$ such that

(1) $\eta_{\mathrm{Spec}\,k} \colon \mathcal{M}(\mathrm{Spec}\,k) \to h_M(\mathrm{Spec}\,k)$ is bijective;
(2) For any scheme N and natural transformation $\nu \colon M \to h_N$, there exists a unique morphism of schemes $f : M \to N$ such that $\nu = h_f \circ \eta$, where $h_f : h_M \to h_N$ is the corresponding natural transformation of presheaves.



Remark 1.4.1. A coarse moduli space for $\mathcal{M}$ is unique up to unique isomorphism, which can be determined by the universal property (2) in the definition.

Proposition 1.4.1. Let (M, η) be a corase moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if

(1) there exists a family $\mathcal{U}$ over M such that $\eta_M(\mathcal{U}) = \mathrm{id}_M$;
(2) for families $\mathcal{F}$ and $\mathcal{G}$ over a scheme S, we have $\mathcal{F} \sim_S \mathcal{G}$ if and only if $\eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

Proof. If (M, η) is a fine moduli space, then (1) and (2) satisfy automatically. Conversely, suppose (M, η) is a coarse moduli space satisfying (1) and (2). In order to show the natural transformation η is a natural isomorphism, it suffices to show that for any scheme S, $\eta_S : \mathcal{M}(S) \to h_M(S)$ is an isomorphism. The condition (2) implies the injectivity. For any morphism $f : S \to M$, the family $\mathcal{F} := f^* \mathcal{U}$ over S satisfies $\eta_S(\mathcal{F}) = f$, and this shows the surjectivity. □

Lemma 1.4.1. Let $\mathcal{M}$ be a moduli problem and suppose there exists a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_1$ for all $s \neq 0$ and $\mathcal{F}_0 \not\sim \mathcal{F}_1$. Then for any scheme M and natural transformation $\eta \colon \mathcal{M} \to h_M$, we have $\eta_{\mathbb{A}^1}(\mathcal{F}) \colon \mathbb{A}^1 \to M$ is constant. In particular, there is no coarse moduli space for this moduli problem.

Proof. Suppose there is a natural transformation $\eta\colon \mathcal{M} \to h_M$. Then η sends the family $\mathcal{F}$ over $\mathbb{A}^1$ to a morphism $f\colon \mathbb{A}^1 \to M$. For any $s\colon \operatorname{Spec} k \to \mathbb{A}^1$, the functoriality implies $f \circ s = \eta_{\operatorname{Spec} k}(\mathcal{F}_s)$

$$\begin{array}{ccc} \mathcal{F} \in \mathcal{M}(\mathbb{A}^1) & \xrightarrow{\eta_{\mathbb{A}^1}} & h_M(\mathbb{A}^1) \ni f \\ \downarrow & & \downarrow \\ \mathcal{F}_s := s^*\mathcal{F} \in \mathcal{M}(\operatorname{Spec} k) & \xrightarrow{\eta_{\operatorname{Spec} k}} & h_M(\operatorname{Spec} k) \ni f \circ s. \end{array}$$

Since for $s \neq 0$, $\mathcal{F}_s = \mathcal{F}_1 \in \mathcal{M}(\operatorname{Spec} k)$, so that $f|_{\mathbb{A}^1 \setminus \{0\}}$ is constant. Let $m\colon \operatorname{Spec} k \to M$ be the k-valued point corresponding to the equivalent class for $\mathcal{F}_1$ under η. Since M is a scheme of finite type, the k-valued points of M are closed, and thus their preimages must also be closed. Then, as $\mathbb{A}^1 \setminus \{0\} \subseteq f^{-1}(m)$, the closure $\mathbb{A}^1$ of $\mathbb{A}^1 \setminus \{0\}$ must also be contained in $f^{-1}(m)$. In other words, f is the constant map from $\mathbb{A}^1$ to the k-valued point m of M. This shows $\eta_{\operatorname{Spec} k}\colon \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is not a bijection, since $\mathcal{F}_0 \neq \mathcal{F}_1$ in $\mathcal{M}(\operatorname{Spec} k)$, but these non-equivalent objects correspond to the same k-point m in M. □

Above lemma provides a pathological behavior, which is called *jumping phenomenon*, for a moduli problem not to admit a corase moduli space.

Example 1.4.1. *Consider the moduli problem of classifying endomorphisms of a n-dimensional k-vector space. To be explicit, a family over a scheme S is a rank n vector bundle $\mathcal{F}$ with an endomorphism ϕ, and $(\mathcal{F}, \phi) \sim_S (\mathcal{G}, \phi')$ if there exists an isomorphism $h\colon \mathcal{F} \to \mathcal{G}$ such that $h \circ \phi = \phi' \circ h$.*

For $n \geq 2$, we can construct families which exhibit the jumping phenomenon. For example, let $n = 2$. Then consider the family over $\mathbb{A}^1$ given by $\mathcal{O}_{\mathbb{A}^1}^{\oplus 2}, \phi)$, where for $s \in \mathbb{A}^1$,

$$\phi_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

For $s, t \neq 0$, these matrices are similar and thus $\phi_s \sim \phi_t$. However, $\phi_0 \not\sim \phi_1$ as these two matrices have different Jordan form.

2. Algebraic stack

2.1. Motivation and definition.

Definition 2.1.1. A *stack* $\mathcal{M}$ is a sheaf of groupoids:

$$\mathcal{M}\colon \mathrm{Sch} \to \mathrm{Groupoids},$$

that is, an assignment

(1) for any scheme T, $\mathcal{M}(T)$ is a category in which all morphisms are isomorphisms.
(2) for any morphism $f\colon X \to Y$, a functor $f^*\colon \mathcal{M}(Y) \to \mathcal{M}(X)$.
(3) for any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, a natural transformation $\varphi_{f,g}\colon f^* \circ g^* \Longrightarrow (g \circ f)^*$. These natural transformations have to be associative for composition, in particular we assume this natural transformation to be the identity, if one of the f,g is identity,

satisfying the following gluing conditions:

(a) Given a covering[2] $\{U_i\}$ of T, objects $\mathcal{E}_i \in \mathcal{M}(U_i)$ and isomorphisms $\varphi_{ij}\colon \mathcal{E}_i|_{U_i\cap U_j} \to \mathcal{E}_j|_{U_i\cap U_j}$ satisfying a cocycle condition on three-fold intersections, there exists an object $\mathcal{E} \in \mathcal{M}(T)$, unique up to isomorphisms, together with isomorphisms $\psi_i\colon \mathcal{E}|_{U_i} \to \mathcal{E}_i$ such that $\varphi_{ij} = \psi_j \circ \psi_i^{-1}$.
(b) Given a covering $\{U_i\}$ of T, objects $\mathcal{E},\mathcal{F} \in \mathcal{M}(T)$ and morphisms $\varphi_i\colon \mathcal{E}|_{U_i} \to \mathcal{F}|_{U_i}$ such that $\varphi|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$, there is a unique morphism $\varphi\colon \mathcal{E} \to \mathcal{F}$ such that $\varphi|_{U_i} = \varphi_i$.

Example 2.1.1. *Let G be an affine algebraic group. Then*

$$BG(T) = \{\textit{principal } G\textit{-bundles on } T\}$$

is the classifying stack of G.

Example 2.1.2. *Let X be a scheme. Then $\underline{X}(T) := \mathrm{Hom}(T,X)$ is a stack, where $\mathrm{Hom}(T,X)$ is considered as a category in which the only morphisms are identities. The pullback functor f^* for $S \to T$ is given by composition with f. Such a stack is called a representable stack.*

Example 2.1.3. *Let X be a scheme and G be an algebraic group acting on X. Then the* quotient stack $[X/G]$ *is defined by*

$$[X/G](T) := \left\{ \begin{array}{ccc} P & \xrightarrow{g} & X \\ {\scriptstyle p}\big\downarrow & & \\ T & & \end{array} : P \to T \textit{ is a } G\textit{-bundle, } P \to X \textit{is a } G\textit{-equivariant map} \right\}.$$

Morphisms in this category are isomorphisms of G-bundles commuting with the maps to X.

[2]A covering means one of the following choices: In complex geometry we use the analytic topology. Otherwise we use either the étale topology or the fppf topology. In this case, the intersection $U_i \cap U_j$ has to be defined as $U_i \times_X U_j$.

Remark 2.1.1. The quotient stack $[X/G]$ should catch the properties of the quotient space "X/G", which may not exist in the category of schemes. Suppose there exists a scheme X/G together with a G-bundle map $X \to X/G$. In this case, any diagram in $[X/G](T)$ defines a map $\overline{g}\colon T \to X/G$ as follows:

$$\begin{array}{ccc} P & \xrightarrow{g} & X \\ {\scriptstyle p}\downarrow & & \downarrow \\ T & \overset{\overline{g}}{\dashrightarrow} & X/G, \end{array}$$

and P is canonically isomorphic to the pullback of the G-bundle $g^*X = X \times_{X/G} T$ over T. Thus the category $[X/G](T)$ is canonically equivalent to the set $X/G(T)$, which is considered as a category in which the only morphisms are the identities of elements.

Lemma 2.1.1 (Yoneda lemma)**.** Let $\mathcal{M}$ be a stack. Then for any scheme T there is a natural equivalence of categories:

$$\mathrm{Mor}_{\mathrm{Stacks}}(\underline{T}, \mathcal{M}) \cong \mathcal{M}(T).$$

Let G be an affine algebraic group and BG be the classifying stack. Let $\mathrm{pt} = \mathrm{Spec}\, k$ be a point and $\mathcal{E}$ be a G-bundle on a scheme X. By Lemma 2.1.1, $\mathcal{E}$ defines a morphism $F_{\mathcal{E}}\colon \underline{X} \to BG$ and the trivial bundle defines a morphism $\mathrm{triv}\colon \mathrm{pt} \to BG$:

$$\begin{array}{ccc} & & \mathrm{pt} \\ & & \downarrow{\scriptstyle \mathrm{triv}} \\ X & \xrightarrow{F_{\mathcal{E}}} & BG. \end{array}$$

Let's compute the fiber product of the above diagram.

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