Abelian variety

Lectured by

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Preface

Motivation and plan. Abelian varieties are special Calabi-Yau varieties. They're "linear", but extremely rich in geometry and arithmetic. For arithmtic, think about BSD conjecture for elliptic curves (which are 1-dimensional abelian varieties) over $\mathbb{Q}$; For geometry, think about Hodge conjecture for abelian varieties over $\mathbb{C}$.

This course is oriented around geometries of abelian varieties, so we basically concern only abelian varieties over an algebraically closed field with characteristic zero. Here is the outline of the course:

(1) Basic structures, [Mum70, Chapter I];
(2) Duality, [Mum70, Chapter II];
(3) Further structures;
(4) Fourier-Mukai transform, [Huy06];
(5) Classifications, [Mum70, Chapter III];
(6) Hodge conjecture for abelian varieties, [Gor97, Mar25, DMOS82];

1. Basic structures

1.1. **Introductions.** One of the most basic and important examples in algebraic geometry is the projective space $\mathbb{P}^n$, which is obtained from projecti. We know that $\pi_i(\mathbb{P}^n) = \{e\}$ for $i \geq 0$ and its Hodge number is given by

$$H^{p,q}(\mathbb{P}^n) \cong \begin{cases} \mathbb{C}, & p = q \\ 0, & \text{otherwise.} \end{cases}$$

There is also an explicit description of the Chow group of $\mathbb{P}^n$ by the standard hyperplane section $[H]$. To be precisely, we have

$$[H^p] \in \mathrm{CH}^p(\mathbb{P}^n) \cong H^{p,p}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z},$$

so the Hodge conjecture holds for $\mathbb{P}^n$ trivially. Also, there are some good chracterizations of $\mathbb{P}^n$:

Theorem 1.1.1 (Mori)**.** Let X be a complex projective manifold with ample tangent bundle. Then X is biholomorphic to $\mathbb{P}^n$.

Theorem 1.1.2 (Yau)**.** Let X be a complex Kähler manifold. If X is homeomorphic to $\mathbb{P}^n$, then X is biholomorphic to $\mathbb{P}^n$.

Yet, we still have another Hartshorne's conjecture, which is still unknown.

Conjecture 1.1.1. Any rank two (holomorphic) vector bundle on $\mathbb{P}^n$ is split for $n \geq 6$.

By Serre's construction, it's equivalent to the following conjecture:

Conjecture 1.1.2. Any smooth codimension two subvariety in $\mathbb{P}^n$ is a complete intersection.

Another important example coming from vector space is elliptic curve. By definition an elliptic curve is the quotient of $\mathbb{C}$ by a lattice, that is, $E = \mathbb{C}/\mathbb{Z}^2$. In higher dimensional cases, simimlar construction provides the complex torus, which is the wedge product of S^1 topologically. However, a surprising result is that for almost all lattices L in $\mathbb{C}^g$, the complex torus $X = \mathbb{C}^g/L$ is not algebraic, that is, there does not exist an algebraic variety $Y/\mathbb{C}$ such that $Y_{an} \cong X$ as complex manifold.

By Chow's theorem, X is algebraic if and only if $X \hookrightarrow \mathbb{P}^n$ for some $n \in \mathbb{Z}_{>0}$, and by Kodaira's embedding theorem, X is algebraic if and only if there exists a positive line bundle on X. In fact, we shall prove the following beautiful result:

Theorem 1.1.3 (Lefschetz)**.** The complex torus $X = \mathbb{C}^g/L$ is algebraic if and only if there exists a positive definite Hermitian metric h on $\mathbb{C}^g$ such that $\mathrm{Im}h(L \times L) \subseteq \mathbb{Z}$.

A cheap way to define abelian variety is to define abelian variety as the complex torus which is algebraic.

1.2. **Hodge structures.** Let X be a compact complex manifold of Kähler type[3]. Then there is the following Hodge decomposition

$$H^k(X,\mathbb{Z}) \cong \bigoplus_{p+q=k} H^{p,q}(X),$$

such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

For the complex torus case, there is additional description on its de Rham cohomology $H^k(X,\mathbb{Z})$. Suppose $X = \mathbb{C}^g/L$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{\pi} & \mathbb{C}^g/L \\ \cong \big| & & \big| \cong \\ T_0X = V & \xrightarrow{\exp} & X. \end{array}$$

This implies that $\pi_1(X) = L$ and thus $H^1(X,\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z}) = L^*$.

If we forget the complex structure, topologically we have $X \cong (S^1)^{2g}$. Then

$$\begin{array}{ccc} H^k(X,\mathbb{Z}) & \xleftarrow{\cong} & \bigwedge^k H^1(X,\mathbb{Z}) \\ \cong\uparrow & & \uparrow\cong \\ H^k((S^1)^{2g},\mathbb{Z}) & \xleftarrow[\cong]{} & \bigwedge^k H^1((S^1)^{2g},\mathbb{Z}). \end{array}$$

In other words, the k-th cohomology is determined by the 1-st cohomology group $H^1(X,\mathbb{Z})$.

In order to compute the Dolbeault cohomology, we equip $X = \mathbb{C}^g/L$ with a Kähler metric ω. Then by the theory of harmonic forms, there is an isomorphism

$$\mathscr{H}^{p,q}(X) = \{\Delta_{\mathrm{d}}(\alpha) = 0 \mid \alpha \in \mathscr{A}^{p,q}(X)\} \cong H^{p,q}(X).$$

Since $X = \mathbb{C}^g/L$ is a Lie group, its tangent bundle is trivial. Thus

$$\mathscr{A}^{p,q}(X) = \mathrm{span}_{C^\infty(X)}\{\mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\overline{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{j_q}\},$$

where $\{\mathrm{d}z^1, \ldots, \mathrm{d}z^g\}$ is a basis of $H^0(X,\Omega^1_X)$.

Note that the above isomorphism is independent of the choice of Kähler metric, we choose the standard flat metric, that is, the metric induced by the Euclidean metric on $\mathbb{C}^g$. Suppose $\alpha = \sum_{|I|=p,|J|=q} f_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$. Then

$$\Delta_{\mathrm{d}}(\alpha) = 0 \Longleftrightarrow \Delta f_{IJ} = 0 \Longleftrightarrow f_{IJ} \in \mathbb{C}.$$

This shows the Hodge number of complex torus $X = \mathbb{C}^g/L$ is

$$h^{p,q}(X) = \binom{g}{p} \times \binom{g}{q}.$$

[3]A compact complex manifold is of Kähler, if there exists a Kähler metric ω on X.

1.3. **Line bundles on complex torus.** In this section, we will show how to describe (holomorphic) line bundles on abelian varieties explicitly. Let X be a complex torus defined by V/L, where $V = \mathbb{C}^g$ and $L \subseteq V$ is a lattice. Note that there is a natural projection $\pi\colon V \to X$, and by Oka-Grauert principle[4], every vector bundle on X pull back to trivial bundle on V, since V is contractible and Stein.

Remark 1.3.1. For line bundles, we may prove above fact algebraically by using the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_V \to \mathscr{O}_V^* \to 0.$$

Indeed, since $H^p(V,\mathbb{Z}) = 0$ for $p > 0$ as V is contractible, and $H^p(V,\mathscr{O}_V) = 0$ for $p > 0$ as V is Stein, then by the long exact sequence induced by the exponential sequence, we have $H^1(V,\mathscr{O}_V^*) = 0$, which shows every line bundle on V is trivial.

Although line bundles on V are trivial, there is a L-action on the structure sheaf $\mathscr{O}_V$, and an L-equivariant locally free $\mathscr{O}_V$-module means an $\mathscr{O}_V$-module $\mathscr{F}$ together with $\phi\colon L \times \mathscr{F} \to \mathscr{F}$ such that $\phi_\ell(fs) = \ell^*(f)\phi_\ell(s)$, where $f \in \mathscr{O}_V$, $s \in \mathscr{F}$ and $\ell^*(f)$ is the action of L on $\mathscr{O}_V$.

Then there is one to one correspondence defined by the pullback functor π^*:

$$\{\text{line bundles on } X\} \longleftrightarrow \{L\text{-equivariant locally free } \mathscr{O}_V\text{-module of rank 1}\}.$$

Thus it suffices to classify all L-equivariant locally free $\mathscr{O}_V$-modules of rank 1. Let $\mathscr{L}$ be a line bundle on X and fix an isomorphism $\alpha\colon \pi^*\mathscr{L} \cong \mathscr{O}_V$, then determine ϕ is equivalant to determine a collection $\{\phi_\ell\}_{\ell\in L}$ such that

$$\phi_{\ell_1+\ell_2} = \ell_2^*\phi_{\ell_1} \cdot \phi_{\ell_2},$$

that is, $\{\phi_\ell\}_{\ell\in L}$ satisfies the cocycle condition. For convenience, the set of all $\{\phi_\ell\}_{\ell\in L}$ which satisfy the cocycle condition is denoted by $Z^1(L,H^*)$, where $H = H^0(V,\mathscr{O}_V)$.

There is an equivalant relation $\sim$ on $Z^1(L,H^*)$ defined by $\{\phi_\ell\} \sim \{\phi'_\ell\}$ if and only if there exists $f \in H^*$ such that for all $\ell \in L$, we have

$$\phi'_\ell \cdot \phi_\ell^{-1} = \ell^*(f) \cdot f^{-1},$$

and the quotient group of $Z^1(L,H^*)/_\sim$ is denoted by $H^1(L,H^*)$.

Then it's not difficult to see the pullback functor π^* induces an isomorphism

$$\begin{aligned} H^1(L,H^*) &\xrightarrow{\cong} H^1(X,\mathscr{O}_X^*) \\ [\{\phi_\ell\}_{\ell\in L}] &\to [\mathscr{L}]. \end{aligned}$$

Now let's introduce how to construct $\{\phi_\ell\}_{\ell\in L}$. Recall that for a Hermitian form h on V, if we write $h = \mathrm{Re}h + \sqrt{-1}\mathrm{Im}h$, then $\mathrm{Re}h$ is symmetric and $E := \mathrm{Im}h$ is alternating. Moreover, E preserves the complex structure of V, that is, $E(\sqrt{-1}x, \sqrt{-1}y) = E(x,y)$ for all $x,y \in V$.

[4]In complex geometry, the Oka-Grauert principle states that over complex manifolds which are Stein manifolds, the non-abelian cohomology-classification of holomorphic vector bundles coincides with that of topological vector bundles.

Suppose we're given a Hermitian form h satisfying the *integrality condition*

$$E\colon L\times L\to\mathbb{Z}.$$

Lemma 1.3.1.

(1) There exists $\alpha\colon L\to \mathrm{U}(1)$ such that for any $\ell_1,\ell_2\in L$, we have

$$\frac{\alpha(\ell_1+\ell_2)}{\alpha(\ell_1)\cdot\alpha(\ell_2)}=e^{\sqrt{-1}\pi E(\ell_1,\ell_2)}\in\{\pm1\}.$$

(2) For $\ell\in L$, if we define

$$\phi_\ell(z)=\alpha(\ell)\cdot e^{\pi h(z,\ell)+\frac{1}{2}\pi h(\ell,\ell)}\in H^*,$$

then $\{\phi_\ell\}\in Z^1(L,H^*)$.

(3) There is a commutative diagram

$$\begin{array}{ccc} [\mathscr{L}]\in H^1(X,\mathscr{O}_X^*) & \xrightarrow{\ \delta\ } & H^2(X,\mathbb{Z}) \\ {\scriptstyle\pi^*}\downarrow & & \downarrow{\scriptstyle\pi^*} \\ [\{\phi_\ell\}_{\ell\in L}]\in H^1(L,H^*) & \xrightarrow{\ \delta\ } & H^2(L,\mathbb{Z}), \end{array}$$

such that $c_1(\mathscr{L})=E$ under the identification $H^2(X,\mathbb{Z})\cong\bigwedge^2L^*$, where $\mathscr{L}$ is the line bundle corresponding to $\{\phi_\ell\}_{\ell\in L}$.

Proof. For (1). Suppose that the rank of L is 2 and take a basis $\{e,f\}$ of L. Then define a map

$$\delta\colon L\to\mathbb{R}$$
$$ne+mf\mapsto\frac{1}{2}nmE(e,f).$$

A direct computation shows that for any $\ell_1,\ell_2\in L$,

$$\delta(\ell_1+\ell_2)-\delta(\ell_1)-\delta(\ell_2)\equiv\frac{1}{2}E(\ell_1,\ell_2)\pmod 1. \tag{1.1}$$

Then we may define $\alpha=e^{2\pi\sqrt{-1}\delta}\colon L\to\mathrm{U}(1)$.

In the general case, we may do induction on the rank of L, or we simply find a symplectic basis of L, denoted by $\{e_1,f_1,e_2,f_2,\dots,e_g,f_g\}$ such that $L=\bigoplus_{i=1}^g L_i$ is an orthogonal decomposition with respect to E, where $L_i=\mathrm{span}_{\mathbb{Z}}\{e_i,f_i\}$. Then a direct computation yields that $\delta\colon L\to\mathbb{R}$ defined by

$$\delta\left(\sum_{i=1}^g(n_ie_i+m_if_i)\right)=\frac{1}{2}\sum_{i=1}^g n_im_iE(e_i,f_i)$$

satisfy (1.1), and we may define $\alpha=e^{2\pi\sqrt{-1}\delta}\colon L\to\mathrm{U}(1)$.

For (2). It follows from direct computation.

For (3). By the exponential sequence

$$0\to\mathbb{Z}\to\mathscr{O}_V\to\mathscr{O}_V^*\to0,$$

there is the following short exact sequence

$$0\to\mathbb{Z}\to H\to H^*\to0, \tag{1.2}$$

since $H^1(V,\mathbb{Z})=0$. Moreover, since V is contractible and Stein, we have $H^i(V,\mathscr{O}_V^*)=0$ for $i \geq 1$. Thus by Appendix to §2 of [Mum70], we get natural isomorphisms as vertical maps

$$\begin{array}{ccc} H^1(X,\mathscr{O}_X^*) & \xrightarrow{\delta} & H^2(X,\mathbb{Z}) \\ \cong \big\| & & \big\| \cong \\ H^1(L,H^*) & \xrightarrow{\delta} & H^2(L,\mathbb{Z}), \end{array}$$

and the commutativity can be checked by using a small open covering of X.

By the commutativity of the diagram, in order to compute the first Chern class of $\mathscr{L}$ corresponding to $\{\phi_\ell\}_{\ell\in L} \in Z^1(L,H^*)$, it suffices to compute $\delta\big(\{\phi_\ell\}_{\ell\in L}\big)$. By the short exact sequence (1.2), we have $Z^1(L,H) \twoheadrightarrow Z^1(L,H^*)$, that is, there exists $\{f_\ell\}_{\ell\in L} \in Z^1(L,H)$ such that $\exp(2\pi\sqrt{-1}f_\ell)=\phi_\ell$. For $\{f_\ell\}_{\ell\in L}$, we have

$$\delta(f_\ell)(\ell_1,\ell_2)(z) = f_{\ell_2}(z+\ell_1) - f_{\ell_1+\ell_2}(z) + f_{\ell_1}(z) \in \mathbb{Z}.$$

Then use the following fact

$$\begin{array}{ccccccc} Z^2(L,\mathbb{Z}) & \xrightarrow{A} & \operatorname{Hom}(\wedge^2 L,\mathbb{Z}) & \xrightarrow{\cong} & \wedge^2 L^* & \xrightarrow{\cong} & H^2(X,\mathbb{Z}) \\ \big\downarrow & & & \nearrow_{\cong} & & & \\ H^2(L,\mathbb{Z}), & & & & & & \end{array}$$

where for $F \in Z^2(L,\mathbb{Z})$, we have $A(F)(\ell_1,\ell_2) := F(\ell_1,\ell_2) - F(\ell_2,\ell_1)$.

Thus we get

$$\begin{aligned} \delta(\{\phi_\ell\}_{\ell\in L})(\ell_1,\ell_2) =& f_{\ell_2}(z+\ell_1) - f_{\ell_1+\ell_2}(z) + f_{\ell_1}(z) - f_{\ell_1}(z+\ell_2) + f_{\ell_1+\ell_2}(z) - f_{\ell_2}(z) \\ =& f_{\ell_2}(z+\ell_1) - f_{\ell_1}(z+\ell_2) + f_{\ell_1}(z) - f_{\ell_2}(z) \\ =& \frac{1}{2\pi\sqrt{-1}}\log\alpha(\ell_2) + \frac{1}{2\pi\sqrt{-1}}\left(\pi h(z+\ell_1,\ell_2) + \frac{1}{2}\pi h(\ell_2,\ell_1)\right) \\ & - \frac{1}{2\pi\sqrt{-1}}\log\alpha(\ell_1) - \frac{1}{2\pi\sqrt{-1}}\left(\pi h(z+\ell_2,\ell_1) + \frac{1}{2}\pi h(\ell_1,\ell_2)\right) \\ & + \frac{1}{2\pi\sqrt{-1}}\log\alpha(\ell_1) + \frac{1}{2\pi\sqrt{-1}}\left(\pi h(z,\ell_1) + \frac{\pi}{h}(\ell_1,\ell_2)\right) \\ & - \frac{1}{2\pi\sqrt{-1}}\log\alpha(\ell_2) - \frac{1}{2\pi\sqrt{-1}}\left(\pi h(z,\ell_2) + \frac{1}{2}\pi h(\ell_2,\ell_2)\right) \\ =& \frac{1}{2\sqrt{-1}}(h(\ell_1,\ell_2) - h(\ell_2,\ell_1)) \\ =& E(\ell_1,\ell_2). \end{aligned}$$

□

Notation 1.3.1. Since the construction of $\{\phi_\ell\}_{\ell\in L}$ depends on Hermitian metric h and α, we write $\mathscr{L}(h,\alpha)$ to denote the line bundle determined by h and α.

Lemma 1.3.2.

$$\mathscr{L}(h_1,\alpha_1)\otimes\mathscr{L}(h_1,\alpha_1)=\mathscr{L}(h_1+h_2,\alpha_1\cdot\alpha_2).$$

Theorem 1.3.1 (Appell-Humbert)**.** Any line bundle on X is isomorphic to a unique $L(H,\alpha)$.

Remark 1.3.2. In other words, if we set

$\mathrm{Herm}^{\mathrm{int}}(V)=\{h : V\times V\to\mathbb{C}\mid h$ is a Hermitian metric and $\mathrm{Im}h$ satisfies the integrable condition$\}$

and

$\widetilde{\mathrm{Herm}}^{\mathrm{int}}(V)=\{(h,\alpha)\mid h\in\mathrm{Herm}^{\mathrm{int}}(V),\ \alpha : L\to\mathrm{U}(1)$ such that $\alpha(\ell_1+\ell_2)=e^{\pi\sqrt{-1}\mathrm{Im}h(\ell_1+\ell_2)}\alpha(\ell_1)\cdot\alpha(\ell_2)\}$,

then we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(L,\mathrm{U}(1)) & \longrightarrow & \widetilde{\mathrm{Herm}}^{\mathrm{int}}(V) & \longrightarrow & \mathrm{Herm}^{\mathrm{int}}(V) & \longrightarrow & 0\\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^{1,1}(X,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

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