Self-dual Equations on Riemann surface

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Contents

1. To readers

It's a note for a reading seminar about self-dual equations on principal bundles, and the main reference is Hitchin's celebrated paper [Hit87]

2. Stability

2.1. **General theory of slope stability.**

2.2. **Stability of rank two Higgs bundle on Riemann surface.** Let X be a compact Riemann surface.

Definition 2.2.1. A Higgs bundle $(\mathcal{E},\theta)$ on X is defined to be *stable* if, for every θ-invariant subbundle $\mathcal{F}\subseteq\mathcal{E}$, we have $\mu(\mathcal{F})<\mu(\mathcal{E})$.

If the Higgs field $\theta=0$, then it reduces to the stability of a vector bundle. However, there exists a stable Higgs bundle $(\mathcal{E},\theta)$, but $\mathcal{E}$ is not stable as a vector bundle.

Example 2.2.1. *Suppose X is a compact Riemann surface with genus $g>1$. Then consider the uniformalizing bundle $\mathcal{V}=K_X^{\frac{1}{2}}\oplus K_X^{-\frac{1}{2}}$ and the Higgs field θ is given by*

$$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Since $K_X^{-\frac{1}{2}}$ is the only θ-invariant subbundle, and it's of negative degree, so $(\mathcal{V},\theta)$ is stable. However, it's clear that $\mathcal{V}$ is not stable as a vector bundle.

An interesting fact is that above phenomenon only happens on compact Riemann surface with genus $g>1$.

Lemma 2.2.1. Let X be a compact Riemann surface of genus $g\le 1$ and $(\mathcal{E},\theta)$ be a semistable Higgs bundle. Then $\mathcal{E}$ is semistable as a vector bundle.

Proof. Suppose $\mathcal{E}$ is not semistable and its Harder-Narasimhan filtration is given by

$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_{s-1}\subset\mathcal{E}_s=\mathcal{E},$$

where the $\mathcal{G}_i=\mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable bundles with $\mu(\mathcal{G}_i)>\mu\left(\mathcal{G}_j\right)$ if $i<j$.

The maximal destabilizer $\mathcal{E}_1=\mathcal{G}_1$ has $\deg(\mathcal{E}_1)>\deg(\mathcal{E})$. Suppose $\theta(\mathcal{E}_1)$ is non-zero and take $\mathcal{E}_\ell$ such that $\theta(\mathcal{E}_1)\subseteq\mathcal{E}_\ell\otimes K_X$ but $\theta(\mathcal{E}_1)\nsubseteq\mathcal{E}_{\ell-1}\otimes K_X$. Then θ induces a non-zero morphism $\mathcal{G}_1\to\mathcal{G}_\ell\otimes K_X$.

Since $g\le 1$, we have $\mu(\mathcal{G}_\ell\otimes K_X)=\mu(\mathcal{G}_\ell)+2g-2\le\mu(\mathcal{G}_\ell)<\mu(\mathcal{G}_1)$, and thus there are no non-zero morphisms unless $\ell=1$, since $\mathcal{G}_i$ are semistable. This shows $\mathcal{E}_1$ is θ-invariant and $(\mathcal{E},\theta)$ is not θ-semistable, a contradiction. □

Example 2.2.2. *There are no stable Higgs bundle of rank two on $\mathbb{P}^1$. Suppose $(\mathcal{E},\theta)$ is a Higgs bundle of rank two. By Grothendieck's classification of vector bundle on $\mathbb{P}^1$, $\mathcal{E}\cong\mathcal{O}_{\mathbb{P}^1}(m)\oplus\mathcal{O}_{\mathbb{P}^1}(n)$, where $m,n\in\mathbb{Z}$. Suppose the Higgs field θ is given by*

$$\begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{pmatrix},$$

where $\theta_{11},\theta_{22}\in H^0(\mathbb{P}^1,K_{\mathbb{P}^1})=H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-2))=0$, $\theta_{12}\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(m-n-2))$ and $\theta_{21}\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(n-m-2))$. Without lose of generality, we may assume $m\ge n$, hence

$\theta_{21} = 0$ and $\mathcal{O}_{\mathbb{P}^1}(m)$ is θ-invariant. However,

$$\mu(\mathcal{O}_{\mathbb{P}^1}(m)) = m > \frac{1}{2}(m+n) = \mu(\mathcal{E}).$$

This shows $(\mathcal{E},\theta)$ is not stable.

Example 2.2.3. *Let $(\mathcal{E},\theta)$ be a rank two stable Higgs bundle on ellipic curve E. Since K_E is trivial, the Higgs field θ are endomorphisms of $\mathcal{E}$, so without lose of generality we may assume $\mathcal{E}$ is indecomposable, otherwise if $\mathcal{E}$ is decomposable, it cannot be stable.*

By Atiyah's classification of vector bundles on ellipic curve ([Ati57, Theorem 5, Theorem 6]*), we know that, after tensoring with a line bundle, $\mathcal{E}$ is equivalent to the non-trivial extension*

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E \to 0$$

defined by $H^1(E,\mathcal{O}_E) = H^0(E,\mathcal{O}_E) \cong \mathbb{C}$, or

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E(1) \to 0$$

defined by $H^1(E,\mathcal{O}_E(-1)) = H^0(E,\mathcal{O}_E(1)) \cong \mathbb{C}$.

In the first case, the distinguished trivial bundle $\mathcal{L} \cong \mathcal{O}$ is invariant by each endomorphism, but $\mu(\mathcal{L}) = 0 = \mu(\mathcal{E})$, which contradicts to the assumption $(\mathcal{E},\theta)$ is stable. In the second case, $\mathcal{E}$ is a stable vector bundle, hence the only endomorphisms are scalars.

Thus the only stable rank two Higgs bundles on ellipic curve are $(\mathcal{E},\theta)$, where $\mathcal{E}$ is the unique indecomposable vector bundle of odd degree and θ is a scalar.

In the case of compact Riemann surface of genus greater than 1, stable Higgs bundles occur with more frequency, such as the uniformalizing Higgs bundle, but there are still restrictions on the holomorphic structure of the underlying vector bundle $\mathcal{E}$.

Proposition 2.2.1. Let X be a compact Riemann surface of genus $g > 1$. A rank two vector bundle $\mathcal{E}$ occurs in a stable Higgs bundle $(\mathcal{E},\theta)$ if and only if one of the following holds:

(1) $\mathcal{E}$ is stable;
(2) $\mathcal{E}$ is semistable and $g > 2$;
(3) If $\mathcal{E}$ is semistable and $g = 2$, then $\mathcal{E} \cong \mathcal{U} \otimes \mathcal{L}$, where $\mathcal{U}$ is either decomposable or an extension of the trivial bundle by itself;
(4) $\mathcal{E}$ is not semistable and $H^0(X,\mathcal{L}^{-2} \otimes K_X \otimes \det\mathcal{E})$ is greater thant 1, where $\mathcal{L}$ is the maximal destabilizer of $\mathcal{E}$.

References

[Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[Hit87] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.