2.6. **Vector bundles on abelian surface.** Let X be an abelian surface with Picard rank one. Let H be an ample generator of $\mathrm{NS}(X)$ and $\widehat{H}$ be the ample generator of $\mathrm{NS}(\widehat{X})$. Then

$$\mathscr{S}(r+dc_1(H)+a\omega) = a - dc_1(\widehat{H}) + r\widehat{\omega},$$

where ω and $\widehat{\omega}$ are fundamental class of X and $\widehat{X}$, respectively.

Proposition 2.6.1. Let $\mathcal{E}$ be a μ-stable sheaf on X of Mukai vector $v(\mathcal{E}) = r - c_1(H) + a\omega$. If $a > 0$, then $\mathcal{E}$ satisfies WIT_2-condition, and $\mathscr{S}^2(\mathcal{E})$ is a μ-stable torsion-free sheaf.

Proof. Firstly we prove that $\mathcal{E}$ is a WIT_2-sheaf. Since $\mathcal{E}$ is μ-stable with negative slope, we know that $\mathrm{Hom}(\mathcal{P}_\xi, \mathcal{E}) = 0$ for all $\xi \in \widehat{X}$. Now we claim that

$$H^1(X, \mathcal{E} \otimes \mathcal{P}_\xi) = 0$$

except for finitely many points $\xi \in \widehat{X}$. Indeed, if $H^1(X, \mathcal{E}\otimes\mathcal{P}_\xi) = \mathrm{Ext}^1(\mathcal{P}_{-\xi}, \mathcal{E}) \neq 0$ for distinct points $\xi = \xi_1, \dots, \xi_n$, then it gives a non-trivial extension

$$0 \to \mathcal{E} \to \mathcal{F} \to \bigoplus_{i=1}^{n} \mathcal{P}_{-\xi_i} \to 0.$$

Now we claim that it is a μ-stable extension. Indeed, if $\mathcal{G} \subseteq \mathcal{F}$ is a subsheaf such that

$$\mu(\mathcal{G}) \geq \mu(\mathcal{F}) = \frac{-H^2}{\mathrm{rk}(\mathcal{F})},$$

then we must have $\mu(\mathcal{G}) \geq 0$, otherwise

$$\mu(\mathcal{G}) \leq \frac{-H^2}{\mathrm{rk}(\mathcal{G})} < \mu(\mathcal{F})$$

since H is an ample generator of $\mathrm{NS}(X)$ and $\mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$.

(1) If $\mu(\mathcal{G}) > 0$, then the composite map $\mathcal{G} \to \mathcal{F} \to \bigoplus_{i=1}^n \mathcal{P}_{-\xi_i}$ is zero, as $\bigoplus_{i=1}^n \mathcal{P}_{-\xi_i}$ is semistable of degree zero, and thus $\mathcal{G}$ is contained in $\mathcal{E}$, which gives a contradiction, since $\mathcal{E}$ is stable and $\mu(\mathcal{E}) < \mu(\mathcal{F})$.
(2) If $\mu(\mathcal{G}) = 0$, without lose of generality, we may assume that the composite map $\mathcal{G} \to \mathcal{F} \to \bigoplus_{i=1}^n \mathcal{P}_{-\xi_i}$ is non-trivial. Moreover, we may assume $\mathcal{G}$ is a μ-stable sheaf, since we may replace $\mathcal{G}$ by its maximal destabilizer and the first term in Jordan-Hölder filtration when necessary. Combine these two facts together we obtain that $\mathcal{G} \cong \mathcal{P}_{-\xi_i}$ for some i, which contradicts to the fact that the extension is non-trivial.

Since it is a stable extension, we have $\langle v(\mathcal{F}), v(\mathcal{F})\rangle = \langle v(\mathcal{E}), v(\mathcal{E})\rangle - 2na \geq 0$. Hence n must satisfy the inequality $n \leq \langle v(\mathcal{E}), v(\mathcal{E})\rangle / 2a$, and thus this completes the proof of claim. As a consequence, we have $\mathscr{S}^0(\mathcal{E}) = 0$ and $\mathscr{S}^1(\mathcal{E})$ is of dimension zero. This means that $\mathscr{S}^1(\mathcal{E})$ is an IT_0-sheaf, but $\widehat{\mathscr{S}}^0\mathscr{S}^1(\mathcal{E}) = 0$, which implies that $\mathscr{S}^1(\mathcal{E}) = 0$. This completes the proof of $\mathcal{E}$ is a WIT_2-sheaf.

Now let's show that $\mathscr{S}^2(\mathcal{E})$ is torsion-free. Let $\mathcal{T}$ be a torsion subsheaf of $\mathscr{S}^2(\mathcal{E})$. Then $\mathcal{T}$ is of dimension zero, as $\mathscr{S}^2(\mathcal{E})$ is locally free in codimension one. Hence $\mathcal{T}$ is an IT_0-sheaf and $\mathscr{S}^0(\mathcal{T})$ is of degree zero. Since $\mathscr{S}^0(\mathcal{T})$ is a subsheaf of $\mathcal{E}$, we must have $\mathscr{S}^0(\mathcal{T}) = 0$, since $\mathcal{E}$ is μ-stable of negative degree. This shows $\mathcal{T} = 0$, that is, $\mathscr{S}^2(\mathcal{E})$ is torsion-free.

Finally, let's show that $\mathscr{S}^2(\mathcal{E})$ is μ-stable. Since $\mathscr{S}^2(\mathcal{E})$ has minimal positive degree, if $\mathscr{S}^2(\mathcal{E})$ is not μ-stable, then there is an exact sequence

$$0 \to \mathcal{A} \to \mathscr{S}^2(\mathcal{E}) \to \mathcal{B} \to 0,$$

where $\mathcal{B}$ is a μ-stable sheaf with degree ≤ 0. By applying the Fourier-Mukai transform, it gives

$$0 \to \widehat{\mathscr{S}}^0(\mathcal{A}) \to (-1_X)^*\mathcal{E} \to \widehat{\mathscr{S}}^0(\mathcal{B}) \to \widehat{\mathscr{S}}^1(\mathcal{A}) \to 0$$

and

$$\widehat{\mathscr{S}}^2(\mathcal{B}) = 0$$
$$\widehat{\mathscr{S}}^1(\mathcal{B}) = \widehat{\mathscr{S}}^2(\mathcal{A}).$$

(1) If $\mathcal{B}$ is of negative degree, then $\mathrm{Hom}(\mathcal{P}_\xi, \mathcal{B}) = 0$ for all $\xi \in \widehat{X}$, and thus $\mathcal{B}$ is a WIT_1-sheaf. But $\mathscr{S}^1\widehat{\mathscr{S}}^1(\mathcal{B}) = \mathscr{S}^1\widehat{\mathscr{S}}^2(\mathcal{A}) = 0$, which implies $\mathcal{B} = 0$.
(2) If $\mathcal{B}$ is of degree zero, then we consider the following cases:
   (a) If $\mathrm{rk}(\mathcal{B}) = 1$, that is, $\mathcal{B} = \mathcal{P}_\xi$ for some $\xi \in \widehat{X}$, then $\widehat{\mathscr{S}}^2(\mathcal{B}) = \mathbb{C}_\xi$, a contradiction.
   (b) If $\mathrm{rk}(\mathcal{B}) \geq 2$, then $h^0(\mathcal{E} \otimes \mathcal{P}_\xi) = h^2(\mathcal{E} \otimes \mathcal{P}_\xi) = 0$ for all $\xi \in \widehat{X}$ since $\mathcal{B}$ is stable. This shows $\mathcal{B}$ is an IT_1-sheaf. However,

   $$\mathscr{S}^1\widehat{\mathscr{S}}^1(\mathcal{B}) = \mathscr{S}^1\widehat{\mathscr{S}}^2(\mathcal{A}) = 0,$$

   which implies $\mathcal{B} = 0$.

This completes the proof.

□

Proposition 2.6.2. Let $\mathcal{E}$ be a μ-stable sheaf on X of Mukai vector $v(\mathcal{E}) = r + c_1(H) + a\omega$. If $a < 0$, then $\mathcal{E}$ satisfies WIT_1-condition, and $\mathscr{S}^1(\mathcal{E})$ is a μ-stable torsion-free sheaf.

Proof. We show that $H^0(X, \mathcal{E} \otimes \mathcal{P}_\xi) = 0$ except for finitely many points $\xi \in \widehat{X}$. Suppose $k_i := h^i(X, \mathcal{E} \otimes \mathcal{P}_{\xi_i}) \neq 0$ for distinct points $\xi_1, \ldots, \xi_n$. We shall consider the evaluation map

$$\phi\colon \bigoplus_{i=1}^n \mathcal{P}_{\xi_i}^\vee \otimes H^0(X, \mathcal{E} \otimes \mathcal{P}_{\xi_i}) \to \mathcal{E}.$$

Without lose of generality, we may assume $\sum_i^n k_i > r$, otherwise we already have $H^0(X, \mathcal{E} \otimes \mathcal{P}_\xi) \neq 0$ for only finitely many $\xi \in \widehat{X}$. By [Yos99, Lemma 2.1], one has ϕ is surjective in codimension 1 and $\ker\phi$ is μ-stable. If we set $b = \dim(\mathrm{coker}\,\phi)$, then the Mukai vector of $\ker\phi$ is

$$v(\ker\phi) = \sum_{i=1}^n k_i - (v(\mathcal{E}) - b\omega).$$

Since $\sum_{i=1}^{n} k_i > r$, we get

$$\begin{aligned}\langle v(\ker\phi), v(\ker\phi)\rangle &= \langle (\sum_i^n k_i - r - c_1(H) + (b-a)\omega, \sum_i^n k_i - r - c_1(H) + (b-a)\omega\rangle \\ &= \langle v(\mathcal{E}), v(\mathcal{E})\rangle + 2(a-b)\sum_i^n k_i + 2br \\ &\leq \langle v(\mathcal{E}), v(\mathcal{E})\rangle + 2a\sum_i k_i.\end{aligned}$$

Since $\langle v(\ker\phi), v(\ker\phi)\rangle \geq 0$, we get

$$\sum_i k_i \leq \frac{\langle v(\mathcal{E}), v(\mathcal{E})\rangle}{-2a}.$$

This completes the proof of finiteness. The base change theorem implies that $\mathscr{S}^0(\mathcal{E})$ is a torsion sheaf of dimension zero. Hence $\mathscr{S}^0(\mathcal{E}) = 0$. By the stability of $\mathcal{E}$ and Serre duality, we have $H^2(X, \mathcal{E} \otimes \mathcal{P}_\xi) = 0$ for all $\xi \in \widehat{X}$. This shows that $\mathcal{E}$ satisfies WIT_1-condition.

Now let's prove that $\mathscr{S}^1(\mathcal{E})$ is torsion-free. Let $\mathcal{T} \subseteq \mathscr{S}^1(\mathcal{E})$ be the torsion subsheaf. By base change theorem, we know that $\mathscr{S}^1(\mathcal{E})$ is locally free on the open subscheme $\{\xi \in \widehat{X} \mid H^0(X, \mathcal{E} \otimes \mathcal{P}_\xi) = 0\}$. Then by above argument we know that $\mathcal{T}$ is supported on finite many points, and thus $\mathcal{T}$ is an IT_0-sheaf. Then

$$0 \to \widehat{\mathscr{S}}^0(\mathcal{T}) \to \widehat{\mathscr{S}}^0\mathscr{S}^1(\mathcal{E}) = 0$$

implies $\mathcal{T} = 0$.

Finally, let's prove $\mathscr{S}^1(\mathcal{E})$ is μ-stable. Suppose that $\mathscr{S}^1(\mathcal{E})$ is not μ-stable. Let

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \cdots \subset \mathcal{F}_s = \mathscr{S}^1(\mathcal{E})$$

be the Harder-Narasimhan filtration. Since $\deg(\mathscr{S}^1(\mathcal{E})) = \deg(\mathcal{E})$ has minimal positive degree, we may choose the integer k such that $\deg(\mathcal{F}_i/\mathcal{F}_{i-1}) > 0$ for $i \leq k$ and $\deg(\mathcal{F}_i/\mathcal{F}_{i-1}) \leq 0$ for $i > k$. In other words, we put $\mathscr{S}^1(\mathcal{E})$ into the following exact sequence

$$0 \to \mathcal{F}_k \to \mathscr{S}^1(\mathcal{E}) \to \mathscr{S}^1(\mathcal{E})/\mathcal{F}_k \to 0.$$

Since $\mathscr{S}^1(\mathcal{E})$ is an IT_1-sheaf, it is clear that $\mathscr{S}^2(\mathscr{S}^1(\mathcal{E})/\mathcal{F}_k) = 0$ and $\mathscr{S}^0(\mathscr{F}_k) = 0$. For any $i \leq k$, the semi-stability of $\mathcal{F}_i/\mathcal{F}_{i-1}$ implies that $\mathscr{S}^2(\mathcal{F}_i/\mathcal{F}_{i-1}) = 0$, and thus $\mathscr{S}^2(\mathcal{F}_k) = 0$. On the other hand, for $i > k$, one can show $\mathscr{S}^0(\mathcal{F}_i/\mathcal{F}_{i-1})$ is of dimension zero. Since $\mathcal{F}_i/\mathcal{F}_{i-1}$ is torsion-free, we have $\mathscr{S}^0(\mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for $i > k$. Hence we conclude that $\mathscr{S}^0(\mathscr{S}^1(\mathcal{E})/\mathcal{F}_k) = 0$.

This shows both $\mathcal{F}_k$ and $\mathscr{S}^1(\mathcal{E})/\mathcal{F}_k$ are WIT_1-sheaves, and we get an exact sequence

$$0 \to \widehat{\mathscr{S}}^1(\mathcal{F}_k) \to \mathcal{E} \to \widehat{\mathscr{S}}^1(\mathscr{S}^1(\mathcal{E})/\mathcal{F}_k) \to 0.$$

Since $\deg(\mathscr{S}^1(\mathcal{F}_k) = \deg(\mathcal{F}_k) > 0$, the μ-stability of $\mathcal{E}$ implies that $\mathrm{rk}(\widehat{\mathscr{S}}^1(\mathcal{F}_k)) = \mathrm{rk}(\mathcal{E})$. Thus $\widehat{\mathscr{S}}^1(\mathscr{S}^1(\mathcal{E})/\mathcal{F}_k)$ is of dimension zero, and thus it is an IT_0-sheaf, a contradiction. $\square$