Recall $\S 1$ Package

$k = \bar{k}$.

- $f : V \to S$ proper flat integral morphism. between k-schemes of finite type.

  F, G : locally free of finite rank on V.

$$\longrightarrow \quad Z \overset{\text{Set}}{=} \{ s \in S \mid \exists \text{ non-zero homomorphism } f_s : F_s \to G_s \}. \qquad Z \subset S \text{ closed}$$

$$\cup \qquad\qquad Z \leadsto \mathrm{Ann}(A)$$

$$W^\circ \overset{\text{Set}}{=} \{ s \in S \mid F_s \cong G_s \}. \qquad A \in \mathrm{Coh}(S)$$

When E : S-simple ($\mathcal{O}_S \xrightarrow{\sim} f_* \mathcal{E}nd_{\mathcal{O}_V}(\mathcal{E})$).

① $W \subseteq Z$ open subscheme. Later with more concrete examples

W = union of connected component of Z.

② W represent the functor

$$\begin{tikzcd} V_T \arrow[r] \arrow[d, "f_T"'] \arrow[dr, phantom, "\Pi"] & V \arrow[d, "f"] \\ T \arrow[r, "\alpha"'] & S \end{tikzcd}$$

$$(\mathrm{Sch}/k) \longrightarrow (\mathrm{Sets})$$

$$T \longmapsto \{ \alpha : T \to S \mid F_T \cong G_T \underset{T}{\otimes} M, \text{ for } M \in \mathrm{Pic}(T) \}.$$

$f : V \to S$	$p_2 : X \times \hat{X} \to \hat{X}$	$p_2 : X \times X \to X$	$p_{23} : X \times \underline{X \times \hat{X}} \to X \times \hat{X}$.
F	$p_1^* E \otimes \mathcal{P}$	$m^* E$	$p_{12}^* m^* E \otimes p_{13}^* \mathcal{P}^{-1}$
G	$p_1^* E$	$p_1^* E$	$p_1^* E$
Z			$\underline{\Phi}(E) \overset{\text{Set}}{=} \{ (x, \hat{x}) \in X \times \hat{X} \mid \exists\, 0 \neq f : t_x^* E \otimes \mathcal{P}_{\hat{x}}^{-1} \to E \}$
W°	$\Sigma^\circ(E) \subseteq \hat{X}$ $= \{ \hat{x} \in \hat{X} \mid E \otimes \mathcal{P}_{\hat{x}} \cong E \}$	$K^\circ(E) \subseteq X$ $= \{ x \in X \mid t_x^* E \cong E \}$	$\Phi^\circ(E) = \{ (x, \hat{x}) \in X \times \hat{X} \mid t_x^* E \otimes \mathcal{P}_{\hat{x}}^{-1} \cong E \}$.
W	$\Sigma(E) \subseteq \hat{X}$	$K(E) \subseteq X$	$\Phi(E) \subseteq X \times \hat{X}$
$F_T \cong G_T \otimes N$	$E_T \otimes \mathcal{P}_f \cong E_T \underset{T}{\otimes} N$, $N \in \mathrm{Pic}(T)$.	$\forall h \in X(S) = \mathrm{Hom}(S, X)$ $h : S \to K(E) \subset X$. $T_h^* E_S \cong E_S \underset{S}{\otimes} N$. $N \in \mathrm{Pic}(S)$	$\forall (h, f) \in (X \times \hat{X})(S) = X(S) \times \hat{X}(S)$ $(h, f) : S \to \Phi(E) \subset X \times \hat{X}$ $T_h^* E_S \otimes \mathcal{P}_f^{-1} \cong E_S \underset{S}{\otimes} N$. $N \in \mathrm{Pic}(S)$

X : ab. var$/k=\bar{k}$. E : v.b. on X.

$\underline{\{t_x^*E \mid x\in X\}}$. $\underline{\{E\otimes P_{\hat{x}} \mid \hat{x}\in\hat{X}\}}$
$\updownarrow$
$P_{\hat{x}}\in \mathrm{Pic}^0(X)$.

$X\times S \xrightarrow{id\times f} X\times\hat{X}$
$(x,s)\mapsto(x,f(s))$.
$P_f := (id\times f)^*P$
P : Poincare bundle on $X\times\hat{X}$

$E_S := p_1^*E$. $p_1 : X\times S\to X$

$T_h : \underset{\sim\sim}{X\times S} \to X\times X\times S\to X\times S$. $T_h^*(E_S)$: v.b. on $X\times S$.
$(x,s)\mapsto(x,h(s),s)$
$(x,x',s)\mapsto(x+x',s)$.

$$\begin{tikzcd}
0 \arrow[dr] & & & & \hat{X} \\
& \Sigma(E) \arrow[dr] & & & \\
& & \Phi(E)\subseteq X\times\hat{X} \arrow[uurr, "p_2"] \arrow[ddrr, "p_1"'] \arrow[ddrr, bend right, "p"'] & & \\
& K(E) \arrow[ur] & & & \\
0 \arrow[ur] & & & & X
\end{tikzcd}$$

$\underset{\sim\sim}{\text{group subsch.}}$
(Later) $\underset{\sim\sim}{\text{closed}}$.

<u>Rmk</u> : $\Phi(E)_{red}\subset X$ closed group subvar. $\Rightarrow$ $\Phi(E)_{red}$: Sub ab. var.

$$\Phi^0(E) = \{(x,\hat{x})\in X\times\hat{X} \mid t_x^*E\otimes P_{\hat{x}}' \cong E\}.$$

$$|\mathrm{Ker}(p_1)| = \{(0,\hat{x})\in X\times\hat{X} \mid E\cong E\otimes P_{\hat{x}}\} = \Sigma^0(E)$$

$|\Sigma(E)| = \Sigma^0(E) \subseteq X[r]$. $\Sigma(E)$: finite sch. $\Rightarrow$ $\dim\Phi(E)\le \dim X = g$.

Prop 3.1 E: v.b. on X. TFAE:

(1) $\forall x \in X$, $t_x^* E \cong E \otimes L$. for some $L \in \mathrm{Pic}(X)$.

(2) $p^\circ : \Phi^\circ(E) \to X$. Surj.

(3) $\dim \Phi^\circ(E) = g$.

Rmk $\boxed{(2) \Leftrightarrow (3) \ \ \text{trivial.} \\ \Updownarrow \\ (1)}$

Def 3.2 E: v.b. on X, E is semi-homogeneous bundle if E satisfies one of the above conditions.

X: ab. var $/k=\bar{k}$.

E: v.b. on X.

$$\Phi^\circ(E) := \{ (x, \hat{x}) \in X \times \hat{X} \mid t_x^* E \cong E \otimes P_{\hat{x}} \}.$$

Last time $\Phi^\circ(E) \subset X \times \hat{X}$ closed subgroup of $X \times \hat{X}$.

Fact locally closed + subgroup $\Rightarrow$ closed subgroup (for topological group).

Constructible + subgroup $\overset{?}{\Rightarrow}$ closed subgroup

$p^\circ : \underbrace{\Phi^\circ(E)}_{\text{ab. subvar. (sm.)}} \subset X \times \hat{X} \to \underset{(\text{sm.})}{X}$ finite group homomorphism.

Miracle flatness

Prop 3.3 E: v.b. on X. $\dim \Phi^0(E) \le \dim X = g$.

P^0: surj. $\Leftrightarrow$ $\dim \Phi^0(E) = g$.

Prop 3.4 E: Simple v.b. on X.

$$\Psi^0(E) := \{ (x, \hat{x}) \in X \times \hat{X} \mid \exists\, 0 \neq f : t_x^* E \to E \otimes P_{\hat{x}} \}.$$

Then $\Phi^0(E)$ = disjoint union of connected components of $\Psi^0(E)$

In particular, $\Phi^{00}(E) \subseteq \Psi^{00}(E)$

Proof. $\left.\begin{array}{l} \Phi^0(E) \text{ is closed in } \Psi^0(E) \\ \Phi^0(E) \underset{\text{open}}{\subseteq} \Psi^0(E) \end{array}\right\} \Rightarrow \checkmark.$ ☒

Thm 5.8. E: Simple v.b. on X. Then TFAE:

(1) $\dim_k H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(E)) = g$

(1') $\dim_k H^j(X, \mathcal{E}nd_{\mathcal{O}_X}(E)) = \binom{g}{j}$. $j = 1, \cdots, g$.

$$\binom{g}{j} = h^j(X, \mathcal{O}_X).$$

$$\begin{array}{ccc} 1) & \Rightarrow & 2) \Leftarrow 4). \\ \Uparrow & & \Downarrow \\ 1') & \Leftarrow & 3) \end{array}$$

Ref §4

(2) E: Semi-homogeneous.

$2) \Rightarrow 3)$

(3) $\mathcal{E}nd_{\mathcal{O}_X}(E)$: homogeneous.

$\forall x \in X$, $t_x^* \mathcal{E}nd_{\mathcal{O}_X}(E) = t_x^* \mathcal{H}om_{\mathcal{O}_X}(E, E)$

(4) $\exists$ isogeny $\pi : Y \to X$, s.t. $E = \pi_* L$, for $L \in \mathrm{Pic}(Y)$.

$= \mathcal{H}om_{\mathcal{O}_X}(t_x^* E, t_x^* E)$

$= \mathcal{H}om_{\mathcal{O}_X}(E \otimes P_{\hat{x}}, E \otimes P_{\hat{x}})$

$\simeq \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

<u>Lemma</u> $\tau : X \to Z$: isogeny. E : Semi-homogeneous bund

$\Rightarrow$. $\tau_* E$: Semi-homogeneous bdl.

<u>Proof</u> <u>Goal</u> $\tau_* E$: Semi-homogeneous bundle

$\Leftarrow \dim \Phi^\circ(\tau_* E) = \dim Z = \dim X$. $(*)$

<u>Claim</u> $\Phi(\tau_* E) \supseteq (\tau \times 1)\big(\underbrace{\Phi^\circ(E) \times_{\hat{X}} Z}_{\dim = \dim X}\big)$. $\Rightarrow$ $(*)$.

$\square$

<u>Prop 3.12</u> $\pi : Y \to X$ isogeny. F : v.b. on Y, $E = \pi_*(F)$.

$\rightsquigarrow$ $(\pi \times 1)\big(\Phi^\circ(F) \times_{\hat{Y}} \hat{X}\big) \subseteq \Phi^\circ(E)$.

<u>Proof</u> $\pi \times 1 : Y \times \hat{X} \longrightarrow X \times \hat{X}$.

$\Phi^\circ(F) \times_{\hat{Y}} \hat{X} \subset (Y \times \hat{Y}) \times_{\hat{Y}} \hat{X}$ $\quad \Phi^\circ(F) \subseteq Y \times \hat{Y}$,

$(y, \hat{y}, \hat{x})$ closed.

$\| \quad Y \times \hat{X}$. $\quad \hat{\pi} : \hat{X} \to \hat{Y}$.

$\begin{cases} (y, \hat{y}) \in \Phi^\circ(F) \\ \hat{y} = \hat{\pi}(\hat{x}) \end{cases}$

$\Phi^\circ(F) \times_{\hat{Y}} \hat{X} = \{ (y, \hat{x}) \in Y \times \hat{X} \mid (y, \hat{\pi}(\hat{x})) \in \Phi^\circ(F) \}$.

$\forall (y, \hat{x}) \in \Phi^\circ(F) \times_{\hat{Y}} \hat{X}$, $(y, \hat{\pi}(\hat{x})) \in \Phi^\circ(F)$

$\rightsquigarrow$ $t_y^* F \cong F \otimes P_{\hat{\pi}(\hat{x})}$

$\Phi^\circ(F) \times_{\hat{Y}} \hat{X} \subset Y \times \hat{X} \xrightarrow{\pi \times 1} X \times \hat{X}$

$$(y, \hat{x}) \longmapsto (\pi(y), \hat{x}) \in \Phi^0(E)$$

$\uparrow$ Check

<u>Check</u>: $t_{\pi(y)}^* E \cong E \otimes P_{\hat{x}}$

$$\begin{tikzcd} Y \arrow[r, "t_y"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\pi"] \\ X \arrow[r, "t_{\pi(y)}"'] & X \end{tikzcd}$$

$\rightsquigarrow \quad t_{\pi(y)}^* \pi_* F = \pi_* t_y^* F$

$= \pi_*(F \otimes P_{\hat{\pi}(\hat{x})}) \qquad P_{\hat{\pi}(\hat{x})} = \pi^* P_{\hat{x}}$

$= \pi_*(F \otimes \pi^* P_{\hat{x}})$

$= \pi_* F \otimes P_{\hat{x}} \qquad \Rightarrow (\pi(y), \hat{x}) \in \Phi^0(E)$

$\#$

Finally (2) $\Rightarrow$ (4)

① §2. $\exists$ isogeny $\pi : Y \to X$. & E' : v.b. on Y s.t

$$\Sigma(E') = 0 \qquad E \cong \pi_* E'.$$

② If E simple semi-homogeneous, So is E' (<u>Ref</u> Prop 5.4(2)).

③ <u>Lemma 5.7</u> E: simple semi-homogeneous bdl. $\Sigma(E) = 0$.

Then E is l.b.

<u>Proof</u>. $\underset{\overset{\parallel}{0}}{\Sigma(E)} \to \Phi(E) \xrightarrow[p]{\sim} X.$

isom.

$\boxed{p^* E \cong L^{\oplus r}} \quad \boxed{r = 1}$

$\Rightarrow E \cong L.$

Lemma 3.6 $p^*E \cong L^{\oplus r}$. $r = \mathrm{rk}(E)$. $(f,h): S \to \Phi(E) \subseteq X \times \hat{X}$. $\to \hat{X}$, $h = p \to X$

Prof. $\Phi(E) \rightsquigarrow \underbrace{T_h^*(E_S)}$ $\cong E_S \otimes \underbrace{P_S \otimes N}_{S.}$
L

$$\begin{array}{ccc} X \times S & \xrightarrow{T_h} & X \times S \\ \cup & & \\ 0 \times S & \longrightarrow & (h(s), s) \\ (0, s) & \longmapsto & \end{array}$$

$|_{0 \times S}$

$E_S = p_1^* E$. $p_1: X \times S \to X$

$$T_h^*(E_S)\big|_{\{0\} \times S} \cong h^* E$$

$$E_S\big|_{\{0\}} \cong \mathcal{O}_S^{\oplus r} \quad r = \mathrm{rk}\, E.$$

$$\Rightarrow \quad h^* E = \mathcal{O}_S^{\oplus r} \otimes \underbrace{L|_S}_{\text{l.b.}}.$$

$S = \Phi^0(E)$. ✓. ☒

Conj. Simple $= \pi_*(\text{l.b.})$. $\pi: Y \to X$.

Takemoto ∃ 3. X: bi-elliptic surface (hyperelliptic surf.).

- In some case X is an abelian scheme.
- ∃ simple v.b. E on X. ($\Delta(E) = 4c_2(E) - c_1^2(E) = 0$
rk $= 2$.

$E \neq \pi_* L$. $\forall\ \pi: Y \to X$ étale

bi-elliptic
X
section ↓ elliptic sm. morphism
E.
elliptic curve

& L: l.b. on Y. $\square$