Semi-homogeneity

①

Let $f: V \to S$ proper, flat, integral

$\mathcal{G}, \mathcal{F}$ locally free over V, then.

$\exists$ coherent sheaf $\mathcal{A}$, s.t $\forall$ quasi-coh $\overset{\mathcal{O}_S\text{-modul}}{\mathcal{M}}$

$$f_*(\mathcal{H}om_{\mathcal{O}_V}(\mathcal{G}, \mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{M}) \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{A}, \mathcal{M})$$

$I = \mathrm{Ann}(\mathcal{A})$, Z closed subscheme given by I

$|Z| = \{ s \in S \mid \exists$ non-zero homomorphism $\mathcal{G}_s \to \mathcal{F}_s \}$

$$W = \{ s \in S \mid \mathcal{G}_s \cong \mathcal{F}_s \}$$
$$\cap\!\!\shortmid$$
$$Z$$

① constructible set

② If $\mathcal{G}$ is S-simple, then $W \subseteq Z$ open.

Moreover, open subscheme W represents

$(Sch) \to (Set)$

$T \longmapsto \{ s: T \to S \mid \underbrace{\mathcal{F}_T \cong \mathcal{G}_T \otimes_{\mathcal{O}_V} N}_{\text{on } V_T}, N \text{ is line bundle on } V \}$

$$\begin{tikzcd} & X \times \hat{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \hat{X} \end{tikzcd}$$

(abelian scheme)?

$$\begin{tikzcd} X \arrow[d] \\ S \arrow[u, bend right, "e"'] \end{tikzcd}$$

Let $\mathcal{E}$ be a locally free sheaf on X

$$\mathcal{F} = p_1^* \mathcal{E} \otimes P_X \qquad \mathcal{G} = p_1^* \mathcal{E}$$

Then $\exists$ coherent sheaf $\mathcal{A}$ on $\hat{X}$, s.t.

$$p_{2*}\left(p_1^* \mathcal{E}nd\,\mathcal{E} \otimes P_X \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{U}\right) \cong \mathcal{H}om_{\mathcal{O}_{\hat{X}}}(\mathcal{A}, \mathcal{U})$$

$$\operatorname{supp} \mathcal{A} = \{ \hat{x} \in \hat{X} \mid \exists \text{ non-zero hom, } p_1^*\mathcal{E}|_{X \times \{\hat{x}\}} \to (p_1^* \mathcal{E} \otimes P_X)|_{X \times \{\hat{x}\}} \}$$

$$\mathcal{E} \to \mathcal{E} \otimes P_{\hat{x}}$$

$$\underset{\substack{\uparrow \\ \text{set}}}{\Sigma^0(\mathcal{E})} = \{ \hat{x} \in \hat{X} \mid \mathcal{E} \cong \mathcal{E} \otimes P_{\hat{x}} \} \subseteq \text{r-torsion pts of } \hat{X}$$

(discrete)

scheme structure on $\Sigma^0(\mathcal{E})$

If we assume $\mathcal{E}$ is simple,

then $\Sigma^0(\mathcal{E}) \underset{\substack{\uparrow \\ \text{open}}}{\subseteq} \operatorname{supp}(\mathcal{A})$

$\Sigma(\mathcal{E})$ is the maximal subscheme s.t.

$$\mathcal{E} \cong \mathcal{E} \otimes P_{\hat{x}}$$

$\Sigma(\mathcal{E})$ represents

$$T \longrightarrow \{ s : T \to \hat{X} \mid \mathcal{E}_T \otimes P_f \cong \mathcal{E}_T \otimes_T \mathcal{L} \}$$

Remk:

$\Sigma^0(\mathcal{E}) = |\Sigma(\mathcal{E})|$

and $\Sigma(\mathcal{E})$ is equipped with the maximal scheme structure

(in the sense of universal property)

Prop:

$\mathcal{E}$ is simple on X, s.t $\Sigma(\mathcal{E}) \neq 0$,

then there exists a non-trivial isogeny

$\lambda : Y \to X$, and a simple v.b., $\mathcal{F}$, s.t

$$\mathcal{E} = \lambda_* \mathcal{F}$$

Cor:

$\mathcal{E}$ is simple on X, s.t $\Sigma(\mathcal{E}) \neq 0$,

then there exists a non-trivial isogeny $\lambda : Y \to X$

and a simple $\mathcal{F}$, s.t $\mathcal{E} = \lambda_* \mathcal{F}$, and $\Sigma(\mathcal{F}) = \{0\}$

Pf: $\Sigma(\mathcal{E})$ is contained in r-torsion pts.

and $rk(\mathcal{E}) = \deg \pi \times rk(\mathcal{F})$.

$\Rightarrow \Sigma(\mathcal{F})$ is contained in $\frac{r}{\deg \pi}$ - torsion pts.

$\cdots \subseteq \frac{r}{\deg \pi}$ - torsions $\subsetneq$ r-torsion

finite steps.

Pf of prop $/ \mathbb{C}$

$\Sigma(\mathcal{E}) \neq 0$, take $G \subseteq \Sigma(\mathcal{E}) \subset \hat{X}$ to be a simple subgroup

Then G is cyclic of order ℓ

Consider

$$\varphi : \hat{X} \to \hat{X}/G \quad \text{projection. is an isogeny}$$

Consider the dual isogeny.

$$\pi : Y \to X \qquad \operatorname{Ker} \pi = \hat{G}$$

(over $\mathbb{C}$, $\pi : Y \to X$ is an unramified cyclic cover)

We want to prove $\exists\, \mathcal{F}$ on Y, s.t.

$$\mathcal{E} = \pi_* \mathcal{F}$$

$$\begin{tikzcd} \xi \arrow[d] \\ \xi \arrow[r] & \xi\otimes J \arrow[r] & \xi\otimes J^2 \arrow[r] & \xi\otimes J^3 \end{tikzcd}$$

(1) $G \curvearrowright \boxed{\mathrm{End}(\pi^*\xi)} = \mathrm{Hom}(\pi^*\xi, \pi^*\xi)$

$$= \mathrm{Hom}(\xi, \pi_*\pi^*\xi)$$

By proj $\quad \pi_*\pi^*\xi \cong \xi\otimes\pi_*\mathcal{O}_Y = \xi^{\otimes \ell}$

By cyclic cover $\quad \pi_*\mathcal{O}_Y = \bigoplus_{i=0}^{\ell-1} J^i$

$$= \bigoplus_{i=0}^{\ell-1} \mathrm{Hom}(\xi, \xi\otimes J^i) = \bigoplus_{i=0}^{\ell-1} \mathrm{Hom}(\xi,\xi) \cong \mathbb{C}[T]/(T^\ell - 1)$$

Claim: $\xi\otimes J^i \cong \xi$.

$$\begin{tikzcd} p_1^*\xi\otimes P|_{X\times G} \cong p_1^*\xi \arrow[d, "p_{1*}"] \\ \xi\otimes p_{1*}(P|_{X\times G}) \cong \xi\otimes p_{1*}\mathcal{O}_{X\times G} \cong \xi^{\otimes\ell} \end{tikzcd}$$

// Fact.

$$\xi\otimes\pi_*\mathcal{O}_Y \cong \xi^{\otimes\ell}$$

$$\pi^* q \cong q_1 \oplus \cdots \oplus q_\ell$$

q_i is simple, since

$$\ell = \mathrm{Hom}(\pi^* q, \pi^* q) = \sum_{1 \le i,j \le \ell} \mathrm{Hom}(q_i, q_j) \ge \ell.$$

$$\pi_* \pi^* q \cong \underbrace{\pi_* q_1} \oplus \cdots \oplus \underbrace{\pi_* q_\ell}$$

$$\cong q^{\oplus \ell}$$

$\Downarrow$ JH fil for simple bundle

$$\pi_* q_i \cong q$$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S & S \arrow[u, bend right, "e"'] \end{tikzcd}$$

$f_s : Y_s \to X_s$ isogeny

$$\Phi^\circ(q) = \{(x, \hat{x}) \in X \times \hat{X} \mid t_x^* q \cong q \otimes P_{\hat{x}}\}.$$

$X \times X \times \hat{X}$

$m : X \times X \to X$

$(s, t) \to s+t.$

$$\mathcal{F} = P_{12}^* \circ m^* q \otimes P_{13}^* P_X^{-1}, \quad \mathcal{G} = P_1^* q$$

$$\mathcal{F}\big|_{X\times\{x\}\times\{\hat{x}\}} = \left(p_{12}^{*}\circ m^{*}\xi \otimes P_{13}^{*} P_{x}^{-1}\right)\Big|_{X\times\{x\}\times\{\hat{x}\}}$$

$$\parallel$$

$$t^{*}\xi\otimes P_{\hat{x}}^{-1}$$

$$\underbrace{i^{*}\circ (m}\circ p_{12})^{*}\xi \simeq p_1^{*}\circ t_x^{*}\xi \simeq t_x^{*}\xi$$

$$\begin{tikzcd}
X\times\{x\}\times\{\hat{x}\} \arrow[r, "i"] \arrow[d, "p_1"] & X\times X\times\hat{X} \arrow[d, "m\circ p_{12}"] \\
X \arrow[r, "t_x"] & X
\end{tikzcd}$$

$\Phi(\xi)$ is the maximal subschem $\subseteq X\times\hat{X}$

s.t. $\mathcal{F}\big|_{\Phi(\xi)} \simeq \mathcal{G}\big|_{\Phi(\xi)}$

$$X\times\hat{X}$$
$$\cup$$
$$p:\ \Phi(\xi) \longrightarrow X$$
$$(x,\hat{x}) \longmapsto x$$

$$\operatorname{Ker} p = p^{-1}(0) = \{\hat{x}\in\hat{X} \mid t_0^{*}\xi \simeq \xi\otimes P_{\hat{x}}\}$$
$$= \Sigma(\xi)$$

Thm

semi-homo + simple

$$\Downarrow$$

$$\xi = \lambda * \lambda$$