§1 Simple vector bundle. All schemes are of finite type over S.

- $f: V \to S$: proper flat integral, F, G : locally free sheaves on V.

$\rightsquigarrow$ $\exists\, A \in \mathrm{Coh}(S)$, s.t. $\forall\, M \in \mathrm{QCoh}(S)$.

$$f_*\left(\mathcal{H}om_{\mathcal{O}_V}(G, F) \otimes_{\mathcal{O}_S} f^*M\right) \underset{\text{natural isom.}}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{O}_S}(A, M).$$

- $\alpha: T \to S$: affine.

$$\begin{tikzcd} V_T \arrow[r] \arrow[d, "f_T"'] \arrow[dr, phantom, "\mathrm{II}"] & V \arrow[d, "f"] \\ T \arrow[r, "\alpha"'] & S \end{tikzcd}$$

$\rightsquigarrow$ $(f_T)_*\left(\underbrace{\mathcal{H}om_{\mathcal{O}_{V_T}}(G_T, F_T)}_{}\right) = (f_T)_*\left(\mathcal{H}om_{\mathcal{O}_V}(G, F) \otimes_{\mathcal{O}_S} \mathcal{O}_T\right)$

$\Vert \rightsquigarrow$ Cartesian diagram.

$\mathcal{H}om_{\mathcal{O}_V}(G, F) \otimes_{\mathcal{O}_S} \mathcal{O}_T$,

$\mathcal{O}_T \in \mathrm{QCoh}(\mathcal{O}_S)$.

$$\begin{tikzcd} M_T & M \in \mathrm{Coh}(C) \arrow[l] \\ B \otimes_A C & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$\mathrm{QCoh}(A)$

$B \otimes_A C$: f.g.

$\cong \mathcal{H}om_{\mathcal{O}_S}(A, \mathcal{O}_T)$

$M_T := M \otimes_C (C \otimes_A B)$

$\cong M \otimes_A B$.

$Z \subseteq S$: closed subscheme of S. $\longleftrightarrow$ $I_Z = \mathrm{Ann}(A) \subseteq \mathcal{O}_S$.

<u>Lemma 1.4</u> $\alpha: T \to S$ morphism. If $F_T \cong G_T \otimes_{\mathcal{O}_T} N$ for some $N \in \mathrm{Pic}(T)$, then α can factor through Z, i.e. $\alpha: T \to Z \to S$

- $|Z| = \mathrm{Supp}\, A = \{ s \in S \mid \exists \text{ non-zero homomorphism } \varphi: G_s \to F_s \}$.

as sets.

$f_* \mathcal{H}om_{\mathcal{O}_{V_T}}(G_T, F_T) \cong \mathcal{H}om_{\mathcal{O}_S}(A, \mathcal{O}_T)$

$\cup|$

$W := \{ s \in S \mid G_s \cong F_s \}$.

<u>Prop 1.5</u> W: Constructible set.

<u>Proof</u> Consider the functor

$$\mathcal{F} : (Sch/S) \longrightarrow (Sets)^{op.}$$
$$T \longmapsto \mathrm{Isom}_{\mathcal{O}_{V_T}}(G_T, F_T)$$

is representable by an open subscheme Y of $\mathbb{V}(A)$

$$\begin{tikzcd} Y \arrow[r, phantom, "\subseteq"] \arrow[dr, "\beta"'] & \mathbb{V}(A) \arrow[d] \\ & S \end{tikzcd}$$

$\leadsto$ $W = \beta(Y)$. Constructible set. $\square$

$\mathcal{F}(T) = \mathrm{Hom}_{Sch/S}(T, Y)$

$\forall s \in S$. $T = \mathrm{Spec}\, \mathcal{O}_{S,s}$. $\mathcal{F}(\mathrm{Spec}\,\mathcal{O}_{S,s}) = \mathrm{Hom}_{(Sch/S)}(\mathrm{Spec}\,\mathcal{O}_{S,s}, Y)$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \| $

$\mathrm{Isom}_{\mathcal{O}_{V_s}}(G_s, F_s)$.

If $s \in W$, $\mathrm{Isom}_{\mathcal{O}_{V_s}}(G_s, F_s) \neq 0 \iff \mathrm{Hom}_{(Sch/S)}(\mathrm{Spec}\,\mathcal{O}_{S,s}, Y) \neq 0$

$$\begin{tikzcd} \mathrm{Spec}\,\mathcal{O}_{S,s} \arrow[r] \arrow[dr] & Y \arrow[d, "\beta"] \\ & S \end{tikzcd} \qquad s \mapsto s \in S$$

$s \in \beta(Y) \Rightarrow W \subseteq \beta(Y)$

G : S-simple. $\left(\mathcal{O}_S \xrightarrow{\sim} f_*(\mathcal{E}nd_{\mathcal{O}_V}(G)), \qquad f : V \to S : \text{proper, flat, integral} \right)$

<u>e.g.</u> $S = \mathrm{Spec}\, k$, $k = \bar{k}$, $\quad \mathrm{End}_{\mathcal{O}_V}(G) = H^0(V, \mathcal{E}nd_{\mathcal{O}_V}(G)) = k$.

G : simple v.b. on V.

<u>Lemma 1.6.</u> $\forall\ w \in W$, $\exists\ w \in \underset{\text{open}}{U \subseteq} Z_1$ s.t. $A|_U \cong \mathcal{O}_Z|_U$

Proof Recall $\alpha: T \to S$. affine.

$$\begin{tikzcd} V_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "f"] \\ T \arrow[r, "\alpha"'] & S \end{tikzcd}$$

$$f_*\left(\mathcal{H}om_{\mathcal{O}_V}(G, F) \otimes \mathcal{O}_T\right) \cong \mathcal{H}om_{\mathcal{O}_S}(A, \mathcal{O}_T).$$

$H^0(-)$ $\rightsquigarrow$

$$\mathrm{Hom}_{\mathcal{O}_{V_T}}(G_T, F_T) = \mathrm{Hom}_{\mathcal{O}_S}(A, \mathcal{O}_T)$$

Taking $T = \mathrm{Spec}\, k(s)$ $\rightsquigarrow$

$$\mathrm{Hom}_{\mathcal{O}_{V_s}}(G_s, F_s) = \mathrm{Hom}_{\mathcal{O}_S}(A, k(s))$$

$V_s := V \otimes_{\mathcal{O}_S} \mathrm{Spec}\, k(s)$

$$= \mathrm{Hom}_{k(s)}(A \otimes k(s), k(s))$$

$$= (A \otimes_{\mathcal{O}_S} k(s))^\vee.$$

G : S-simple $\Rightarrow$ $\forall\, \omega \in W$, G_ω : simple on V_ω

$$\rightsquigarrow \left[A \otimes_{\mathcal{O}_S} k(\omega)\right]^\vee = \mathrm{Hom}_{\mathcal{O}_{V_\omega}}(G_\omega, G_\omega) \cong k(\omega).$$

$$\left(W = \{\omega \in S \mid G_\omega \cong F_\omega\}. \right)$$

$$\rightsquigarrow \quad \dim_{k(\omega)} A \otimes k(\omega) = 1.$$

Nakayama's Lemma $\Rightarrow$ $\exists\ \omega \in \tilde{U} \underset{\text{open}}{\subseteq} S$ and $I \underset{\text{ideal}}{\subseteq} \mathcal{O}_{\tilde{U}}$, s.t.

$$\underbrace{A|_{\tilde{U}} = \mathcal{O}_{\tilde{U}}/I}.$$

?? $\boxed{\underbrace{I_Z|_{\tilde{U}} = I}}$ $\Rightarrow$ $A|_{\tilde{U}} = A|_{Z \cap \tilde{U}} = A|_U = \mathcal{O}_U/I_Z = \mathcal{O}_{Z \cap U}$

$U = Z \cap \tilde{U}$ $\quad$ $\mathrm{Supp}\, A = |Z|$ $\quad$ $I_Z = \mathrm{Ann}(A)$. $\quad$ $U = \tilde{U} \cap Z$

$I_Z = \mathrm{Ann}(A)$

$$Z = \mathrm{Spec}_S(\mathcal{O}_S/I_Z).$$

$$A|_{\tilde{U}} = \mathcal{O}_{\tilde{U}}/I. \Rightarrow I_Z|_{\tilde{U}} = \mathrm{Ann}(A)|_{\tilde{U}}$$

$$= \mathrm{Ann}(\mathcal{O}_{\tilde{U}}/I) = I.$$

$\blacksquare$

Prop 1.7 $W \subseteq Z$: open subset.

Proof. $\forall\, w \in W$, $\exists\, w \in U \subseteq Z$, s.t. $A|_U \cong \mathcal{O}_U$

$\mathrm{Hom}_{\mathcal{O}_U}(G_U, F_U) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_U}(A|_U, \mathcal{O}_U) \cong \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U)$.

$\longrightarrow k(w') \cong \mathrm{Hom}_{\mathcal{O}_{w'}}(G_{w'}, F_{w'})$, $\forall\, w' \in U \subseteq Z$

$\Rightarrow \exists\, \varphi \in \mathrm{Hom}_{\mathcal{O}_U}(G_U, F_U)$, s.t. $\forall\, s \in U$, $\varphi|_s : G_s \xrightarrow{\sim} F_s$. $s \in W$.

$\Rightarrow U \subseteq W$. $\square$

$\rightsquigarrow$ W has an open subscheme structure.

? Where is the constructible set ?

Rmk W = largest subsch s.t. F and G are isom. to each other

Thm 1.8. Consider the functor

$$\mathcal{F} : (Sch/S) \longrightarrow (Sets)^{op}$$

$$T \longmapsto \{\alpha : T \to S \mid F_T \cong G_T \otimes_{\mathcal{O}_T} M \text{ for some } M \in \mathrm{Pic}(T)\}.$$

Then W represent $\mathcal{F}$.

Proof " $\forall\, T \in (Sch/S)$, $\mathcal{F}(T) = \mathrm{Hom}_{(Sch/S)}(T, W)$ ".

$$\begin{tikzcd} V_W \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "f"] \\ W \subset Z & \subset S \end{tikzcd}$$

Put. $L = (f_W)_* \mathcal{H}om_{\mathcal{O}_{V_W}}(G_W, F_W)$.

Claim L : invertible sheaf on W.

$\rightsquigarrow$ $\lambda : G_W \otimes L \xrightarrow{\sim} F_W$ $\Longrightarrow$ $(W \subseteq S) \in \mathcal{F}(W)$

$\rightsquigarrow$ $\forall\, (\alpha : T \to S) \in \mathcal{F}(T)$, $\alpha : T \longrightarrow Z \subseteq S$, $\to W$

We want. $\alpha(T) \subseteq W$.

$G_T \otimes_{\mathcal{O}_T} M \xrightarrow{\sim} F_T$

$W = \{ s \in S \mid G_s \cong F_s \}$. $\square$

$Z = \{ \quad G_s \to F_s \text{ non-zero} \}$

$\exists$ N : invertible on Z, s.t. $G_Z \otimes N \to F_Z$.

$$\begin{tikzcd} \alpha^*(G_Z \otimes N) \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & \alpha^* F_Z \arrow[d, phantom, "\wr"] \\ G_T \otimes M \arrow[r, "\sim"] & F_T \end{tikzcd} \Rightarrow G_Z \otimes N \xrightarrow{\sim} F_Z.$$

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§2 The group scheme $\Sigma(E)$

$k = \bar{k}$, char $= p \geq 0$. X : ab. var. of dim $= g / k$. $\mathcal{P}_X$: normalized Poincaré bundle.

E: v.b. of rk $= r$ on X.

$\hat{X} = \mathrm{Pic}^0(X)$, $\hat{x} \mapsto P_{\hat{x}}$

Def $\Sigma^0(E) = \{ \hat{x} \in \hat{X} \mid E \otimes P_{\hat{x}} \cong E \}$.

- $\Sigma^0(E) \subseteq \hat{X}[r] := \ker(X \xrightarrow{\times r} X)$

- Taking $F := p_1^* E \otimes \mathcal{P}$, $G := p_1^* E$.

$$\begin{tikzcd} & X \times \hat{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \hat{X} \end{tikzcd}$$

$\exists$ $A \in \mathrm{Coh}(\hat{X})$, s.t. $\forall M \in \mathrm{QCoh}(\hat{X})$,

$$(p_2)_* \Big(\underbrace{p_1^* \mathcal{E}nd_{\mathcal{O}_X}(E) \otimes \mathcal{P}}_{\shortparallel} \otimes_{\mathcal{O}_{\hat{X}}} M \Big) \cong \mathcal{H}om_{\mathcal{O}_{\hat{X}}}(A, M).$$

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_{X \times \hat{X}}}(G, F) &= \mathcal{H}om_{\mathcal{O}_{X \times \hat{X}}}(p_1^* E, p_1^* E \otimes \mathcal{P}) \\ &= \mathcal{H}om_{\mathcal{O}_{X \times \hat{X}}}(p_1^* E, p_1^* E) \otimes_{\mathcal{O}_{X \times \hat{X}}} \mathcal{P} \\ &= p_1^* \mathcal{E}nd_{\mathcal{O}_X}(E) \otimes_{\mathcal{O}_{X \times \hat{X}}} \mathcal{P}. \end{aligned}$$

Now E: simple. $\rightsquigarrow$ $G := p_1^* E$: $\hat{X}$-simple

Def $\Sigma(E) :=$ the largest subscheme s.t. F and G isom. to each other.

Thm $\forall S \in (Sch/k)$,

$$S \to \Sigma(E) \subset \hat{X} \qquad P_f := (id\times f)^* P$$

$$X \times S \xrightarrow{id \times f} X \times \hat{X}$$

$$\mathrm{Hom}_{Sch/k}(S, \Sigma(E)) = \mathcal{F}(S) = \{ f \in \hat{X}(S) \mid \underbrace{\underset{F_S}{E_S \otimes P_f} \cong \underset{G_S}{E_S \otimes_{\mathcal{O}_S} N} \text{ for } N : \text{invertible on } S} \}$$

$$\begin{tikzcd} X \times S \arrow[r, "id \times f"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & X \times \hat{X} \arrow[d, "p_2"] \\ S \arrow[r, "f"'] & \hat{X} \end{tikzcd}$$

$$E_S = (id \times f)^* (p_1^* E) = (p_1^* E)_S$$

- $\Sigma(E) \subset \hat{X}$ group subscheme.

Check $\mathcal{F}(S)$ is a group. $\forall f, g \in \mathcal{F}(S)$.

$\rightsquigarrow$ $f, g \in \hat{X}(S)$, s.t. $\exists N_f, N_g$: invertible on S, s.t.

$$E_S \otimes P_f \cong E_S \otimes N_f, \quad E_S \otimes P_g \cong E_S \otimes N_g,$$

Claim $E_S \otimes P_{f+g} = E_S \otimes_{\mathcal{O}_S} (N_f \otimes N_g) \Rightarrow f+g \in \mathcal{F}(S)$.

$$\Big(\underbrace{(1_X \times m)^* P = \pi_{12}^* P \otimes \pi_{13}^* P} \Big) \rightsquigarrow \text{Seesaw}$$

$m : \hat{X} \times \hat{X} \to \hat{X}$

$$1_X \times m : X \times \hat{X} \times \hat{X} \xrightarrow{1 \times m} X \times \hat{X}$$

$$\begin{tikzcd} & X \times \hat{X} \times \hat{X} \arrow[dl, "\pi_{12}"'] \arrow[dr] & \\ X \times \hat{X} & & X \times \hat{X} \end{tikzcd}$$

$\square$

- $\Sigma(E) \subseteq \hat{X}[r]$ $\rightsquigarrow$ Σ : finite group. subsch.

$\boxtimes$