Homogeneous bundle on AV

Let X be an abelian var$/_{k=\bar{k}}$, $\hat{X}$ be its dual abelian var

A coherent sheaf $\mathcal{Q}$ is homogeneous if $t_x^*\mathcal{Q} \cong \mathcal{Q}$

$$\{\text{homogeneous sheaf}\}$$
$$\updownarrow$$
$$\left\{ \bigoplus_{i=1}^{n} P_i \otimes U_i \;\middle|\; P_i \in \mathrm{Pic}^0(X),\ \underset{\substack{\uparrow \\ \mathcal{O}_X \text{ by } \mathcal{O}_X \text{ sucessive extention}}}{U_i} \text{ is unipotent bundle} \right\}$$
$$\updownarrow$$
$$\{\text{coherent sheaf with finite support on } \hat{X}\}$$

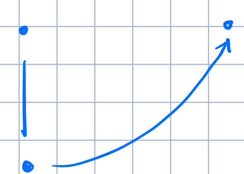
Ex

Let $\mathbb{C}_{\hat{x}}$ be the skyscraper sheaf on $\hat{X}$, supported at $\{\hat{x}\}$

$$\begin{array}{ccc} & X \times \hat{X} & \\ \swarrow & & \searrow \\ X & & \hat{X} \end{array}$$

$$\Phi(\mathbb{C}_{\hat{x}}) = p_{1*}\left(P_X \otimes \underbrace{p_2^*\mathbb{C}_{\hat{x}}}\right)$$
$$= p_{1*}\left(P_X\big|_{X\times\{\hat{x}\}}\right)$$
$$= P_{\hat{x}} \in \mathrm{Pic}^0(X)$$

$$\Phi \circ \Phi = (-1)_X^*[-g]$$



Prop $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ exact seq of sheaves

s.t $\Phi(\mathcal{E}), \Phi(\mathcal{F}), \Phi(\mathcal{G})$ concentrate at i-th. degree

($\mathcal{E}, \mathcal{F}, \mathcal{G}$, satisfy WIT_i - condition

week index theorem

IT_i - condition

except $i=0$

① $\Phi(\mathcal{E}), \Phi(\mathcal{F}), \Phi(\mathcal{G})$, are localy free

② $H^k(\mathcal{E} \otimes P_\xi) = 0$, for all $\xi \in \mathrm{Pic}^0(X)$

cohomology base

except $k = i$

$\chi(\mathcal{E}) = h^i(\mathcal{E}\otimes P_\xi)$ is constant

Pf $\quad 0\to \mathcal{E}\to \mathcal{F}\to \mathcal{G}\to 0$

$$\downarrow$$

$$0\to p_2^*\mathcal{E}\to p_2^*\mathcal{F}\to p_2^*\mathcal{G}\to 0$$

$$\downarrow$$

$$0\to R^ip_{1*}(p_2^*\mathcal{E}\otimes P_x)\to R^ip_{1*}(p_2^*\mathcal{F}\otimes P_x)\to R^ip_{1*}(p_2^*\mathcal{G}\otimes P_x)\to 0$$

$\{\bigoplus_{i=1}^n P_i\otimes U_i \mid P_i\in \mathrm{Pic}^0(X),\ U_i$ is unipotent bundle$\}$

O_X by O_X sucessive extention

$\{$coherent sheaf with finite support on $\hat{X}\}$

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$$0\to \mathbb{C}_{\hat{x}}\to \mathcal{E}\to \mathbb{C}_{\hat{x}}\to 0$$

$$\downarrow$$

$$0\to P_{\hat{x}}\to \Phi(\mathcal{E})\to P_{\hat{x}}\to 0$$

$\otimes P_{\hat{x}}'$

$$\Phi(\mathcal{E})\in \underbrace{\mathrm{Ext}^1(P_{\hat{x}},P_{\hat{x}})}\cong \mathrm{Ext}^1(O_X,O_X)$$

Prop

$$\Phi\circ t_x^*\cong \otimes P_{-x}\circ\Phi$$

$$\Phi\circ\otimes P_{\hat{x}}\cong t_{\hat{x}}\circ\Phi$$

$$\begin{tikzcd} & X\times\hat{X} \arrow[loop above, "t_{(x,0)}"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[loop below, "t_x"] & & \hat{X} \end{tikzcd}$$

Pf: $\mathcal{F}^\bullet\in D^b(X)$

$$\Phi(t_x^* \mathcal{F}^\bullet) = Rp_{2*}(P_X \otimes \underbrace{p_1^* \circ t_x^*}_{} \mathcal{F}^\bullet)$$

$$= Rp_{2*}(P_X \otimes t_{(x,0)}^* p_1^* \mathcal{F}^\bullet) \qquad \sim \ \underbrace{P_X \otimes p_2^* P_{-x}}_{}$$

$$= Rp_{2*} \circ t_{(x,0)}^* \left(\boxed{\underbrace{t_{(-x,0)}^* P_X}_{}} \otimes p_1^* \mathcal{F}^\bullet \right)$$

(Seesaw principle

[X, Y, complete. λ line bundle on $X \times Y$

If $\lambda|_{X \times \{y\}}$ is trivial for all y over an open dense subset

of Y, and $\lambda|_{\{x_0\} \times Y}$ is trivial, then λ is trivial]

$$\mathcal{F} = \underbrace{P_X \otimes p_2^* P_{-a}}_{} \cong \underbrace{t_{(-a,0)}^*}_{} P_X = \mathcal{G}$$

$$\mathcal{F}|_{\{x\} \times \hat{X}} = (P_X \otimes p_2^* P_{-a})|_{\{x\} \times \hat{X}} \qquad \mathcal{G}|_{\{x\} \times \hat{X}} = P_X|_{\{x-a\} \times \hat{X}}$$

$$= P_X|_{\{x\} \times \hat{X}} \otimes P_{-a} \qquad\qquad = P_{x-a}$$

$$= P_x \otimes P_{-a}$$

$$= P_{x-a}$$

$$\mathcal{F}|_{X \times \{\hat{0}\}} = \mathcal{O}_X \qquad \mathcal{G}|_{X \times \{\hat{0}\}} = (t_{(x,0)}^* P_X)|_{X \times \{\hat{0}\}}$$

$$= t_{-x}^* \mathcal{O}_X$$

$$= \mathcal{O}_X$$

$$= \underbrace{Rp_{2*} \circ t_{(x,0)}^*}_{} (\underline{p_2^* P_{-x}} \otimes P_X \otimes p_1^* \mathcal{F}^\bullet)$$

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "t_{(x,0)}"] \arrow[d, "p_2"'] & X \times \hat{X} \arrow[d, "p_2"] \\ \hat{X} \arrow[r] & X \end{tikzcd}$$

$$= R p_{2*}\left(p_2^* P_{-x} \otimes P_x \otimes p_1^* \mathcal{F}^\bullet\right)$$

$$= \Phi(\mathcal{F}^\bullet) \otimes P_{-x}$$

Prop

① coherent sheaf $\mathcal{M}$ on $\hat{X}$ s.t. $\mathcal{M} \otimes P_x \cong \mathcal{M}$ for all $x \in X$, then $\mathrm{supp}(\mathcal{M})$ is finite

$\Downarrow$

② $\mathcal{M}^\bullet$ on $\hat{X}$, s.t. $\mathcal{M}^\bullet \otimes P_x \cong \mathcal{M}$, then cohomology sheaf has finite support

Pf: Suppose $\mathrm{supp}(\mathcal{M})$ contains a curve C

$\iota : \hat{C} \xrightarrow[\varphi]{} C \subseteq \hat{X}$ ($\hat{C}$ is not $\mathbb{P}^1$)

normalization

$$\begin{tikzcd} & C^J \arrow[d, "\alpha"] \\ \mathbb{P}^1 \arrow[ur] \arrow[r, "f"] & X \end{tikzcd}$$

simply-connected

$\mathcal{F} := (\iota^* \mathcal{M}) / \mathrm{tor}(\iota^* \mathcal{M})$ bundle on $\hat{C}$

$\mathcal{F} \otimes \iota^* P_x \cong \left(\iota^*(\mathcal{M} \otimes P_x)\right) / \mathrm{tor}\left(\iota^*(\mathcal{M} \otimes P_x)\right)$

$\cong \iota^* \mathcal{M} / \mathrm{tor}(\iota^* \mathcal{M})$

$\cong \mathcal{F}$

take determinant

$$\cancel{\det \mathcal{F}} \otimes \iota^* P_x^{\otimes r} \cong \cancel{\det \mathcal{F}}$$

$$\iota^* P_x^{\otimes r} = \mathcal{O}_{\tilde{C}}$$

that is,

$$\mathrm{Pic}^\circ(\hat{X}) \xrightarrow{\iota^*} \mathrm{Pic}^\circ(\tilde{C}) \xrightarrow{\times r} \mathrm{Pic}^\circ(\tilde{C}) \text{ is zero}$$

$$\begin{tikzcd}
\mathrm{Pic}^\circ(\tilde{C}) \arrow[dr, bend left, leftrightarrow, "\text{dual}"] & \\
J(\tilde{C}) \arrow[r] & \hat{X} \\
\tilde{C} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$\mathrm{Im}\, \iota^* \subseteq r$-torsion points of $\mathrm{Pic}^\circ(\tilde{C})$, contra

Finally

$\mathcal{F}$ homogeneous,

it suffices to prove $\Phi(\mathcal{F})$ is sheaf

Note that $\Phi \circ \Phi = (-1_X)^* [-g]$

there is a spectral seq

$$E_2^{p,q} = \Phi^p \circ \Phi^q(\mathcal{E}) \Longrightarrow E_\infty^{p,q} = \begin{cases} (-1_X)^* \mathcal{E}, & p+q=g \\ 0, & \text{otherwise} \end{cases}$$

$\Phi^q(\mathcal{E})$ has finite support

$\Phi^p \circ \Phi^q(\xi) = 0$, if $p > 0$

E_2-degenerate

$\Phi^g \Phi^0(\xi) \cong (-1_X)^* \xi$, others are zero

So only $\Phi^0(\xi)$ is not zero

other $\Phi^q(\xi) = 0$, $q \neq 0$