I. Overview on Poincaré bundle

universal property

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II. Simple vector bundles on ab var.

"bi-elliptic" ✗

Semi-homo — simple

↑

homogeneous

"Goal" E: simple $\overset{\text{Semi-homogeneous v.b.}}{\wedge}$ on A → ab var,

$\exists\ \pi: B \to A$ & L: l.b. on B, s.t. $E \cong \pi_* L$.

B ↓ ab var

π ↓ isogeny → étale

($\ker \pi$: finite, surjective)

Rmk ① rat'l maps between ab. var. are morphism. Ex. Universal property of Poincaré bundle

② morphisms between ab. var are composition of homomorphism and translation. (rigidity)

Def An abelian variety is a proper group variety$/k$ + multiplication is commutative "abelian" (rigidity)

Char $= 0$ $A^{an} \cong \mathbb{C}^N/\Lambda$

Rmk Char $> \phi$. Char $= 0$. any finite surjective map between abelian var. is étale. "separated"

$$\begin{tikzcd} \mathbb{C}^N \arrow[r, dashed, "\tilde{\pi}"] \arrow[d] & \mathbb{C}^N \arrow[d] \\ B \arrow[r, "\pi"] & A \end{tikzcd}$$

linear

$A = \mathbb{C}^N/\Lambda_A$. $B = \mathbb{C}^N/\Lambda_B$.

$\tilde{\pi}(\Lambda_B) \subseteq \Lambda_A$ $\pi_1(A) \cong \Lambda_A$

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↓ Covering map $\Rightarrow$ étale

↓ $\pi_*(\pi_1(B)) \subseteq \pi_1(A)$. finite index subgroup.

Rmk A: ab. var$/k=\bar{k}$ $\forall\, a \in A(k)$. $t_a: A \longrightarrow A$

$b \longmapsto b+a$.

Rmk π: homomorphism. $\forall b \in B$. $d\pi_b : \boxed{T_b B \longrightarrow T_{\pi(b)} A}$

$\forall b' \in B$. $t_{b'}^*(d\pi_b) = d\pi_{b+b'} : T_{b+b'}B \to T_{\pi(b+b')}A$

$$T_{\pi(b+b')}A = T_{\pi(b)+\pi(b')}A$$

I. Picard scheme and Poincaré bundle.

Picard functor $\mathrm{Pic}_{(X/S)} : (Sch/S) \longrightarrow (Sets)^{op}$ $\qquad X_T := X \times_S T$

$$T \longmapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$$

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

$\mathrm{Pic}^\circ_{(X/S)} : T \longrightarrow \left(\mathrm{Pic}(X_T)/\mathrm{Pic}(T) \right)^\circ$

Thm $f: X \to S$: flat, projective, morphism. s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & S \arrow[d] \\ & \text{var.}/k \end{tikzcd}$$

(1) Every geom. fiber is connected and reduced, and

(2) the irreducible component of any fiber is geometrically irred.

Then $\mathrm{Pic}_{(X/S)}$ can be representable by a scheme $\mathrm{Pic}_{X/S}$, locally of finite type

Moreover, if all fibers are geom. integral, then $\mathrm{Pic}_{X/S}$: separated$/S$.

($k = \bar{k}$, f : sm)

- If $\mathrm{Pic}_{X/S}$ exists. $\forall s \in S$

$\mathrm{Pic}^\circ_{X_s/k(s)}$: Connected component of Pic_{X_s/k_s} containing identity element

$$\mathrm{Pic}_{X_s/k_s} = \mathrm{Pic}_{X/S} \times_S k(s)$$

$$\mathrm{Pic}^\circ_{X/S} := \bigcup_{s \in S} \mathrm{Pic}^\circ_{X_s/k_s}. \quad \text{Set.}$$

Thm X : ~~proper~~ projective var.$/k$, char $=0$. Then

(1) $\mathrm{Pic}_{X/k}$ exists. $\mathrm{Pic}_{\mathbb{P}^1/\mathbb{C}} \cong \mathbb{Z}$: NOT finite type.

(2) The identity component $\mathrm{Pic}^0_{X/k}$: quasi-proj.? + sm. var.

(3) If X is normal, then $\underline{\underline{\mathrm{Pic}^0_{X/k}}}$: proj. and hence is an $\underline{\text{ab. var.}}$

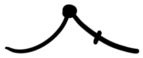
(Fact : ① A proper connect alg. group var. is an ab. var.$/k$

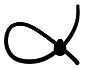
② Ab. var. are always proj.

e.g. $C \subseteq \mathbb{P}^2_{\mathbb{C}}$: cubic curve, then

$\mathrm{Pic}^0_{C/\mathbb{C}} = \Big\{$

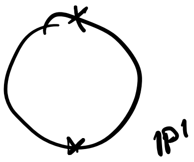
C , if C : Smooth (elliptic) curve

$\mathbb{G}_a$, if  $\leftarrow \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$

$\mathbb{G}_m$, if  nodal

$\uparrow$

$\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^\times \cup \{0\} \cup \{\infty\}$



Universal object

$$\mathrm{Pic}_{(X/S)}(\mathrm{Pic}_{X/S}) = \frac{\mathrm{Pic}(X \times_S \mathrm{Pic}_{X/S})}{\mathrm{Pic}(\mathrm{Pic}_{X/S})} \simeq \mathrm{Hom}(\mathrm{Pic}_{X/S}, \mathrm{Pic}_{X/S}).$$

$$\mathcal{U}_{X/S} \longleftarrow\!\shortmid \ 1_{\mathrm{Pic}_{X/S}}$$

$X=A$, $S=\mathrm{Spec}\, k$ $\quad \hat{A} = \mathrm{Pic}^0_{A/k}$ abelian var "dual".

"Poincaré bundle" $\quad \mathcal{U}_{A/k}\big|_{\mathrm{Pic}^0_{A/k}} =: \mathcal{P}_A$

e.g. $e \in A$ identity element. "normalized"

$e \in \hat{A}$ $\qquad \mathcal{P}_A\big|_{e\times\hat{A}} \cong \mathcal{O}_{\hat{A}}$, $\mathcal{P}_A\big|_{A\times e} \cong \mathcal{O}_A$

- (Universal property) $\forall S \in (\mathrm{Sch}/k)$

$$\frac{\mathrm{Pic}(A \times_k S)}{\mathrm{Pic}(S)} \simeq \mathrm{Hom}(S, \mathrm{Pic}_{A/k})$$

$$\mathcal{L} : \text{lb} \longmapsto \varphi : S \to \mathrm{Pic}_{A/k}$$
$$s \longmapsto \mathcal{L}|_s$$

$$(1_A \times \varphi)^* \mathcal{U}_A \longleftarrow \varphi$$

$1_A \times \varphi : A \times S \to A \times \mathrm{Pic}_{A/k}$

If $\mathcal{L}$: $\underline{\mathcal{L}\big|_{S\times e} = \mathcal{O}_S}$ and $\underline{\forall s \in S,\ \mathcal{L}|_{s\times A} \in \mathrm{Pic}^0(A)}$,

$\mathcal{L} \longmapsto \varphi : S \to \mathrm{Pic}^0_{A/k}$, $s \mapsto \mathcal{L}|_s \in \mathrm{Pic}^0(A)$.

$$\mathcal{L} = (1_A \times \varphi)^* \mathcal{P}_A$$

↪ normalized.

- $\forall\, \alpha \in \hat{A}$, $\quad \mathcal{P}_A|_{A\times\alpha} = \mathcal{P}_\alpha \in \mathrm{Pic}^0(A)$

  $\forall\, a \in A$, $\quad \mathcal{P}_A|_{a\times\hat{A}} = \mathcal{P}_a \in \mathrm{Pic}^0(\hat{A})$.

- L : l.b. on A/k.

$$\begin{tikzcd} & A\times A \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & A \end{tikzcd}$$

$m : A\times A \to A$ multiplication

<u>Mumford bundle</u> on $A\times A$: $\Lambda(L) := m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$

$H^2(A,\mathbb{Z})$

$c_1(t_a^*L\otimes L^{-1}) \in\; = 0$

① $\Lambda(L)|_{A\times e} \cong \mathcal{O}_A$ $\qquad$ ② $\forall a\in A$, $\Lambda(L)|_{a\times A} \cong t_a^*L\otimes L^{-1} \boxed{\in \mathrm{Pic}^0(A)}$

?

$t_a : A \to A$, $b \mapsto b+a$

$$\begin{tikzcd} A\times e \arrow[r, phantom, "\subset"] \arrow[rr, bend left, "p_1"] & A\times A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} a\times A \arrow[r, phantom, "\subset"] \arrow[rr, bend left, "t_a"] & A\times A \arrow[r, "m"] & A \end{tikzcd}$$

$\rightsquigarrow$ $\varphi_L : A \to \mathrm{Pic}_{A/k}$, $\quad a \mapsto [t_a^*L\otimes L^{-1}] = \alpha \in \hat{A}$

$\qquad\qquad\qquad\qquad\qquad\qquad e \mapsto [\mathcal{O}_A] = e \in \hat{A}$

$\rightsquigarrow$ $\varphi_L : A \to \hat{A}$, $\boxed{\text{homomorphism}}$

- $(1\times\varphi_L)^*\mathcal{P}_A = \Lambda(L)$.

- $t_a^*L \cong L^{-1}\otimes \mathcal{P}_\alpha$

$$\boxed{\begin{array}{c} \mathbb{C} \cong \overline{\mathbb{C}(\mathbb{P}^1)} \\ \text{as fields} \end{array}}$$

- $K(L) := \ker(\varphi_L) = \{ a\in A \mid t_a^*L \cong L \}$.

$g : A \to B$ homomorphism. $L = g^*L'$. $B = A/K(L)^0$ L' : <u>non-degenerate</u>

$K(L') \subseteq B$ <u>finite</u>.

- (thm of square)

$$t_{a+b}^*L\otimes L^{-1} = \varphi_L(a+b) = \varphi_L(a)+\varphi_L(b)$$

$$= (t_a^*L\otimes L^{-1}) \otimes (t_b^*L\otimes L^{-1})$$

$$\Rightarrow \quad t_{a+b}^{*}L \otimes L \cong t_a^{*}L \otimes t_b^{*}L.$$

∘ L : $\underline{\text{non-deg.}}$ ($K(L)$: finite).

$$K(L) = \ker(\varphi_L) \;, \quad \varphi_L : A \to \hat{A} = A/K(L).$$

$$\# K(L) = \chi(L)^2 = \deg\Big(\big(\underbrace{ch(L) . td(T_A)}_{\overset{\shortparallel}{1}}\big)_g \Big)^2 = \Big(\frac{L^g}{g!}\Big)^2$$

$\hookrightarrow$ non-degenerate.

$\boxed{g = \dim A}$

$E = g_* L$

$$\Rightarrow \; c_1(E) = \frac{g_* g^* c_1(E)}{\deg g} = \frac{\cancel{r} g_* c_1(L)}{\cancel{\deg g}}$$

$$c_1(g^*E) = \underbrace{r c_1(L)}$$

$\overset{\shortparallel}{g^* c_1(E)}$ ∘

$c_1(E) \rightsquigarrow \underline{c_1(L)}$

$\boxed{L}$

E : v.b on A

$\rightarrow$ $E \otimes P$. $P \in Pic^0(A)$

$\rightarrow$ $t_a^* E$, $a \in A$

$$\Sigma(E) = \{ \alpha \in \hat{A} \mid E \otimes P_\alpha \cong E \} . \subseteq \hat{A}$$

$g : B \to A$

$g^* E = L^{\otimes r}$

$$\boxed{\Phi(E)} = \underline{\{ (a,\alpha) \mid t_a^* E \cong E \otimes P_\alpha \} \subseteq A \times \hat{A}}$$

Step I $\rightsquigarrow$ $\underline{\Sigma(E) = 0}$.

• $\det(E \otimes P_\alpha) = \det E \otimes P_\alpha^r$ $r = rk E$.

$\overset{\shortparallel}{\det E}$ $\Rightarrow$ $P_\alpha^{\otimes r} \cong \mathcal{O}_A$ $\Rightarrow$ $\alpha \in \underline{\hat{A}[r]}$

$\hat{A}[r] := \ker(\hat{A} \xrightarrow{\times r} \hat{A})$.

$\rightsquigarrow$ $\Sigma(E)$: finite.

$\underline{\pi : B \to A}$. isogeny homomorphism $\pi_*(E_1) = E$.

~~$P_1 \neq P_2 \in \ker \hat{\pi}$,~~

~~s.t. $E \otimes P_1 \cong E \otimes P_2$.~~

E : v.b. on A.

$$\pi_*(\pi^* E) = E \otimes \pi_* \mathcal{O}_B .$$

$$= E \otimes \underbrace{\bigoplus_{P \in \ker \hat{\pi}} P} = \bigoplus_{P \in \ker \hat{\pi}} E \otimes P$$

$\rightsquigarrow$ $\hat{\pi} : \hat{A} \to \hat{B}$. isogeny . homomorphism

" $\underline{\text{Exer}}$: (char = 0) $\pi_* \mathcal{O}_B = \bigoplus_{P \in \ker(\hat{\pi})} P$ "