Goal:

Let $E$ be an elliptic curve and $u \in \mathbb{Q}$

$$\mathrm{Vect}(E)_u \longrightarrow \mathrm{Vect}(E)_0 \longleftrightarrow \mathrm{Coh}_{tor}(E)$$

category of s.s bundle with $\frac{c_1(\mathcal{E})}{\mathrm{rk}\,\mathcal{E}} = u$

Fourier–Mukai on abelian variety $X$

$$D^b(X) \to D^b(\hat{X})$$

$$\mathcal{F}^\bullet \mapsto Rp_{2*}(P_X \otimes p_1^* \mathcal{F}^\bullet),$$

$$\begin{tikzcd} & X \times \hat{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \hat{X} \end{tikzcd}$$

For elliptic curve $E$,

$$E \cong \hat{E} := \mathrm{Pic}^0(E) = V/\Gamma$$

(For a general abelian var $X$, $X \to \hat{X} := \mathrm{Pic}^0(X)$ is an isogeny)

(finite, surjective

If there exists a $\underbrace{\text{polarization}}_{\text{(ample line)}}$ $\lambda$,

$$c_1(\lambda) \in H^{1,1}(X) \cap H^2(X, \mathbb{Z}) \qquad \Gamma^* := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$$

$$H^2(X,\mathbb{Z}) \cong \Lambda^2 H^1(X, \mathbb{Z}) \cong \Lambda^2 \Gamma^* \qquad (\Gamma \cong \pi_1(X) = H_1(X, \mathbb{Z}))$$

$c_1(\lambda)$ can be seen as a skew-symmetric form

$$c_1(\lambda) : \Gamma \times \Gamma \to \mathbb{Z}$$

Take a suitable basis of $\Gamma$ (symplectic basis)

$c_1(\lambda)$ is of the form

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & n_1 \\ -n_1 & 0 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} 0 & n_g \\ -n_g & 0 \end{matrix}} \end{pmatrix} \qquad n_1 \mid n_2 \mid \cdots \mid n_g$$

$(n_1, \ldots, n_g)$ is called the type of $\mathcal{L}$

An ample line bundle $\mathcal{L}$ is called a principal polarization, if it's of type $(1, \ldots, 1)$

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FM transform given by the Poincaré bundle

$\Phi : D^b(E) \longrightarrow D^b(E)$

Let $\mathcal{F}$ be a s.s bundle of slope $\mu$

$\Phi(\mathcal{F})$ concentrates at one degree

Lemma

If $\mu \neq 0$, then $\exists\ i$ s.t $H^i(E, \mathcal{F} \otimes P) \neq 0$ for all $P \in \mathrm{Pic}^0(E)$ and for all $j \neq i$, $H^j(E, \mathcal{F} \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(E)$

① proof of the Lemma

② why Lemma implies $\Phi(\mathcal{F})$ concentrates at one degree

① If $\mu < 0$, $H^0(E, \mathcal{F} \otimes P) = \mathrm{Hom}(P^*, \mathcal{F})$

but $P^*$ is of slope $0$, and $\mathcal{F}$ is s.s, of slope $\mu < 0$, $\Rightarrow$ $\mathrm{Hom}(P^*, \mathcal{F}) = 0$

but $H^1(E, \mathcal{F} \otimes P) \neq 0$,

For $\mu > 0$ case, it's the same by Serre duality

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② $\Phi(\mathcal{F}) := Rp_{2*}(\underbrace{P_X \otimes p_1^* \mathcal{F}})$

$$P_X \otimes p_1^* \mathcal{F} \text{ on } X \times \hat{X}$$

and $(P_X \otimes p_1^* \mathcal{F})\big|_{X \times \{\xi\}} = \underbrace{P_\xi \otimes \mathcal{F}}$, $\xi \in \hat{X}$

So, If there is only one $i$ s.t

$$H^i(\mathcal{F}, P_\xi \otimes \mathcal{F}) \neq 0 \text{ for all } \xi \in \hat{X}$$

By cohomology base change,

$R^i p_{2*}(P_X \otimes p_1^* \mathcal{F}) \neq 0$, others are zero

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$\mathcal{F}$, s.s. of slope $\mu < 0$ $\quad \Phi(\mathcal{F})[1]$ is a sheaf

$$\left( R^i(\Phi(\mathcal{F})[1]) = R^{i+1}\Phi(\mathcal{F}) \right)$$

what's the degree, rank of $\Phi(\mathcal{F})[1]$

Lemma A

$$\deg(\Phi(\mathcal{F}^\bullet)) = -\mathrm{rk}(\mathcal{F}^\bullet), \quad \mathrm{rank}(\Phi(\mathcal{F}^\bullet)) = \deg \mathcal{F}^\bullet$$

$P_\xi \quad \mathrm{Pic}^0(X)$

$\xi \quad /\!/$

$\Phi(\mathcal{F}) \in D^b(\hat{X})$

Lemma: For all AV

For any $\mathcal{F} \in D^b(X)$, $\xi \in \hat{X}$, we have

$$R\Gamma(X, \mathcal{F} \otimes P_\xi) = \Phi(\mathcal{F}^\bullet)\big|_{\{\xi\}}$$

Pf:

$$\begin{tikzcd} X \times \{\xi\} \arrow[r] \arrow[d, hook] & \{\xi\} \arrow[d, hook] \\ X \times \hat{X} \arrow[r, "\text{flat}"] & \hat{X} \end{tikzcd}$$

( check ? )

Tor independence

$$\Phi(\mathcal{F}^\bullet)\big|_{\{\xi\}} = Rp_{2*}(P_x \otimes p_1^* \mathcal{F}^\bullet)\big|_{\{\xi\}}$$

$$\overset{?}{=} Rp_{2*}\left((P_x \otimes p_1^* \mathcal{F}^\bullet)\big|_{X\times\{\xi\}}\right)$$

$$= R\Gamma(P_\xi \otimes \mathcal{F}^\bullet) \quad \#$$

proof of Lemma A

Take $\mathcal{F}^\bullet \in D^b(E)$, by above Lemma

$$R\Gamma(E, \mathcal{F}^\bullet) \cong \underbrace{\Phi(\mathcal{F}^\bullet)}\big|_{\{e\}}$$

$$\underbrace{\mathrm{rk}\,\Phi(\mathcal{F}^\bullet)} = \underbrace{\sum (-1)^i \dim R^i\Gamma(E, \mathcal{F}^\bullet)} = \chi(E, \mathcal{F}^\bullet)$$

But by Riemann-Roch,

$$\chi(E, \mathcal{F}^\bullet) = \underbrace{\deg}(\mathcal{F}^\bullet)$$

On the other hand

$$\underbrace{R\Gamma(E, \Phi(\mathcal{F}^\bullet))} = \underbrace{\Phi \circ \Phi(\mathcal{F}^\bullet)}\big|_{\{e\}}$$

$$\| $$

$$(-1)^* \mathcal{F}^\bullet \underbrace{[-1]}\big|_{\{e\}}$$

$$-\mathrm{rk}(\mathcal{F}^\bullet) = \deg \Phi(\mathcal{F}^\bullet)$$

For higher dim case

$$R\Gamma(X, \mathcal{F}^\bullet) \cong \Phi(\mathcal{F}^\bullet)|_{\{e\}}$$

tangent bundle is trivial

$$\mathrm{rk}\,\Phi(\mathcal{F}^\bullet) = \chi(X, \mathcal{F}^\bullet) \overset{\text{HRR}}{=} \int_X \mathrm{ch}(\mathcal{F}^\bullet)\underbrace{\mathrm{Td}(X)}_{=1}$$

$$= \mathrm{ch}_g(\mathcal{F}^\bullet)$$

$$R\Gamma(X, \Phi(\mathcal{F}^\bullet)) = (-1)^* \mathcal{F}^\bullet[-g]|_{\{e\}}$$

$$\mathrm{ch}_g(\Phi(\mathcal{F}^\bullet)) = (-1)^g \mathrm{rk}(\mathcal{F}^\bullet)$$

$$\boxed{PD(\mathrm{ch}_i(\Phi(\mathcal{F}^\bullet))) = (-1)^i \mathrm{ch}_{g-i}(\mathcal{F}^\bullet)}$$

( Fourier-Mukai
Huybrechts )

$$H^{2i}(\hat{X}, \mathbb{Q}) \xrightarrow{PD} H^{2g-2i}(X, \mathbb{Q})$$

$$X = \mathbb{C}^n/\Gamma \rightsquigarrow \hat{X} = \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C})/\hat{\Gamma}$$

Question

$X = B \times F$, $\mathcal{E}$, $(\mathrm{ch}_0(\mathcal{E}), \mathrm{ch}_1(\mathcal{E}), \mathrm{ch}_2(\mathcal{E}))$

Summarize

s.s bundle $\mathcal{F}$, with slope $u \neq 0$

$\Phi$

$\Phi(\mathcal{F})_g$ is semistable bundle of slope $\frac{-1}{u}$

Lemma

If $\mathcal{F}$ is s.s of slope $u \neq 0$, then $\Phi(\mathcal{F})$ is again s.s.

$u \in \mathbb{Q}$

Pf. WLOG, we may assume $\mathcal{F}$ is indecomposable.

( $\mathcal{F} = \oplus \mathcal{F}_i$, claim, each $\mathcal{F}_i$ is of slope $u$ )

Then $\Phi(\mathcal{F})$ is again indecomposable

But on elliptic curve, every indecomposable bundle is s.s (since HN filtration splits)

$$\mathrm{Vect}(E)_u \longrightarrow \mathrm{Vect}(E).$$

① $\mathcal{F} \longmapsto \Phi(\mathcal{F})$

$u \longmapsto -\frac{1}{u}$

② $\mathcal{F} \longmapsto \mathcal{F} \otimes \mathcal{O}_E(e)$

$u \longmapsto u+1$

- $SL_2(\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- $SL_2(\mathbb{Z})$ acts on $\mathbb{P}^1_{\mathbb{Q}}$ transitively

$S \curvearrowright \mathbb{P}^1_{\mathbb{Q}}$ by $u \mapsto \frac{-1}{u}$

$T \curvearrowright \mathbb{P}^1_{\mathbb{Q}}$ by $u \mapsto u+1$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \frac{ax+b}{cx+d}}$$

$$\underbrace{GL_2(\mathbb{Z})} \curvearrowright \mathbb{Z}^2$$

$$\underbrace{\mathbb{P}^1_{\mathbb{Q}}} \cong \mathbb{P}^1_{\mathbb{Z}}$$

$$SL_2(\mathbb{C}) \curvearrowright \mathbb{P}^1_{\mathbb{C}}$$

$$PGL_2(\mathbb{C}) = \underbrace{PSL_2(\mathbb{C})} \twoheadleftarrow SL_2(\mathbb{C}) \hookleftarrow \{\pm 1\} \leftarrow 0$$

$\curvearrowright \mathbb{P}^1_{\mathbb{C}}$

$\{\det A = 1\}$

$$PGL_2(\mathbb{R}) \neq P\underbrace{SL_2}(\mathbb{R})$$

$\curvearrowright \mathbb{P}^1_{\mathbb{R}}$

$\{$s.s. slope 0 on $E\} \longleftrightarrow \{$tor sheaves on $E\}$

Mehta

Atiyah 1956 over curve ✓, AV ✓, general var Biswas

$\{$s.s. vanishing chern characters$\} \longleftrightarrow \{$admitting a (flat) algebraic connection$\}$

over $p$

$\longleftrightarrow \{$homogeneous bundle$\}$

$\longleftrightarrow$ { coherent sheaves with finite support }

$\updownarrow$ ?

{ finite extension of skyscraper sheaves }

Ques:

S.S $\xrightarrow{FM}$ S.S on curve.

How about higher dimension case?

known:

surface case, partial result.