Mukai 78 Semi-homogeneous vector bundles on abelian var.

<u>Goal</u>: Classification on vector bundles / Coherent sheaves on ab. var.

I. Background / Known results

II. Thm (Mukai 78, Mehta/Nori 84) A: ab. var., $\mathcal{E}$: v.b. on A.

Then

$$\underline{\mathcal{E}: \text{Semi-homogeneous}} \iff \underline{\mathcal{E}: \text{Semi-Stable}} + \boxed{\begin{matrix} c(\mathcal{E}) = \left(1 + \frac{c_1(\mathcal{E})}{r}\right)^r \\ r = \operatorname{rank} \mathcal{E}. \end{matrix}}$$

" zero proj. Chern class ".

I. Background & Known results.

Vector bundles on curves.

On $\mathbb{P}^1_k$. k: field

<u>Thm</u> (Grothendieck). E: v.b. of $\operatorname{rk} = r$ on $\mathbb{P}^1$. Then

$\exists!$ integers $a_1 \geq a_2 \geq \cdots \geq a_r$, s.t. $E \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$. $\square$

<u>e.g.</u> $r = 1$. $\{\text{l.b. on } \mathbb{P}^1\}/\sim_{\text{isom.}} = \{\ldots, \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots\}$

unbounded

<u>Recall</u> X: proj. Sch$/k$. A set of isom. classes of coherent sheaves on X is <u>bounded</u> if $\exists\, S$: k-sch. of <u>finite type</u>, $\mathcal{F} \in \operatorname{Coh}(X \times_k S)$, s.t. Given set $\subseteq \{\mathcal{F}|_{X \times \{s\}} \mid s \in S: \text{closed pt}\}$.

Q : $\mathbb{P}^1_{\mathbb{Z}}$? $\mathbb{P}^1_{\mathbb{Q}}$. $\mathbb{P}^1_{\mathbb{F}_p}$.

e.g. $\deg E = \sum_{i=1}^{n} a_i$

$\{$ v.b. on $\mathbb{P}^1$ with $r=2$, $d=0 \} / \sim_{\text{isom.}} = \{ \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k) \}_{k \in \mathbb{Z}}$.

"unbounded".

$\star$ Semistable, Stable (Gieseker).

Slope semistable. Slope stable. $\mu = \frac{\deg}{\mathrm{rk}}$.

$k>0$ $\underline{\mathcal{O}(k)} \oplus \mathcal{O}(-k) \longrightarrow d=0.\ r=2.\ \mu=0.$

$\mathcal{O}(k) \longrightarrow d=k,\ r=1,\ \mu=k>0.$

Stable $k<0$. Semi-stable $k \le 0$.

Def X^n: proj. Sch $/k$. H: ample. $\mathcal{F} \in \mathrm{Coh}(X)$.

$$P_{\mathcal{F}}(m) := \frac{\chi(X, \mathcal{F} \otimes H^{\otimes m})}{H^n} \in \mathbb{Q}[m].$$ Hilbert poly.

$\forall\ \underbrace{P_1, P_2 \in \mathbb{Q}[m]}_{\text{monic}}$, $P_1 \ge P_2$ if $\forall\ m \gg 0$, $P_1(m) \ge P_2(m)$

$\mathcal{F}$: (Gieseker) semistable (stable) if $\forall\ \mathcal{E} \subseteq \mathcal{F}$ and $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(\mathcal{F})$. $P_{\mathcal{E}} \le P_{\mathcal{F}}$. $(P_{\mathcal{E}} < P_{\mathcal{F}})$.

"torsion free"

Slope $\mu(\mathcal{F}) := \dfrac{\deg \mathcal{F}}{\mathrm{rk}\,\mathcal{F}} = \dfrac{c_1(\mathcal{F}).H^{n-1}}{\mathrm{rk}\,\mathcal{F}}$

$\mathcal{F}$: Slope $(H\text{-})$ Semistable if $\forall\ \mathcal{E} \subseteq \mathcal{F}$ and $0 < \mathrm{rk}(\mathcal{E}) < \mathrm{rk}(\mathcal{F})$,
" torsion free " (Stable)

$\mu(\mathcal{E}) \leq \mu(\mathcal{F})$
$(\ \mu(\mathcal{E}) < \mu(\mathcal{F})\ )$.

Slope stable $\Rightarrow$ stable. $\Rightarrow$ Semi-stable $\Rightarrow$ Slope Semi-stable.

On elliptic curves (Sm. genus one curve w/ a pt. fixed (E, e)).

Thm (Atiyah 57)

① $\mathcal{F}$: v.b. on E, $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$, $\mathcal{F}_i$: Semistable.

(On Curve (Gieseker = slope). Harder-Narasimhan filtration

$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subseteq \mathcal{E}_k = \mathcal{F}$. $\mathcal{E}_i \subseteq \mathcal{F}$: saturated $\mathcal{F}/\mathcal{E}_i$: torsion free
$\Updownarrow$ curve
locally free

$\mu(\mathcal{E}_1/\mathcal{E}_0) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu(\mathcal{E}_k/\mathcal{E}_{k-1})$.

$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_2/\mathcal{E}_1 \to 0$ $\quad \mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1)$

$\mathrm{Ext}^1(\mathcal{E}_2/\mathcal{E}_1, \mathcal{E}_1) = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2/\mathcal{E}_1 \otimes \omega_E) = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2/\mathcal{E}_1)$

$\omega_E \cong \mathcal{O}_E$ $\qquad$ $= 0$)

$\mathcal{E}_1, \mathcal{E}_2/\mathcal{E}_1$: Semistable

$\square$

② Fix $\mu \in \mathbb{Q}$

Composition of FM transform

$$\mathrm{Vect}(E)_\mu := \{\text{Semistable on } E \text{ with slope } \mu\} \hookrightarrow \mathrm{Coh}_{\mathrm{tors}}(\hat{E})$$

$\downarrow$ abelian cat. $\qquad$ equiv. of cat.

$\frac{1}{2} \longmapsto$ length $=1$

$\mu = \frac{d}{r} \longmapsto$ length $= \gcd(d,r)$

$\mathcal{E}, \mathcal{F} \in \mathrm{Vect}(E)_\mu$

$0 \to \mathcal{E} \to \mathcal{F}$, Claim $\mathcal{F}/\mathcal{E}$: torsion-free

(Otherwise, $\mathcal{E} \subseteq \mathcal{E}' \subseteq \mathcal{F}$

$\mu(\mathcal{E}') > \mu(\mathcal{E}) = \mu(\mathcal{F}) \Rightarrow\Leftarrow$!)

$\hat{E} \cong E$.

$$\begin{tikzcd} & P_E \arrow[d] & \\ & E \times \hat{E} \arrow[dl] \arrow[dr] & \\ E & & \hat{E} \end{tikzcd}$$

e.g. $\mathcal{F}$: v.b. on E w/ $\deg \mathcal{F} > 0$

Some FM transform $\Phi : \mathrm{Coh}(E) \to \mathrm{Coh}(E)$, $\mathcal{F} \mapsto \Phi(\mathcal{F})$

$$\begin{pmatrix} r(\Phi\mathcal{F}) \\ d(\Phi\mathcal{F}) \end{pmatrix} = \underset{\in SL_2(\mathbb{Z})}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} r(\mathcal{F}) \\ d(\mathcal{F}) \end{pmatrix}$$

If $\begin{pmatrix} r(\mathcal{F}) \\ d(\mathcal{F}) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} r(\Phi\mathcal{F}) \\ d(\Phi\mathcal{F}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$\mu = \frac{1}{2} \longrightarrow \mu = -2 . \xrightarrow{\otimes \mathcal{O}_E(2e)} \underset{\substack{P \\ \in \\ \mathrm{Pic}^0(E)}}{\mu = 0} . \xrightarrow{\text{FM}}$ torsion sheaf on E.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$d(C) = \sum (-1)^i \deg(\mathcal{H}^i C)$

$D^b(E) \longrightarrow D^b(E)$

$\{\underset{0}{P}\} \longmapsto \{\underset{0}{0} \to \underset{1}{T}\}$

$$r(C) = \sum (-1)^i \, \mathrm{rk}(\mathcal{H}^i C).$$

Conclusion

① FM transform induced by $\mathcal{P}_E$

$$\begin{tikzcd} & \mathcal{P}_E : \text{line bundle} \arrow[d] & \\ & E \times \hat{E} \arrow[dl, "p_E"'] \arrow[dr, "p_{\hat{E}}"] & \\ E & & \hat{E} \end{tikzcd}$$

$$R\Phi_E : D^b(E) \longrightarrow D^b(\hat{E})$$
$$\mathcal{F} \longmapsto Rp_{\hat{E}*}(p_E^* \mathcal{F} \otimes \mathcal{P}_E)$$

$$\begin{pmatrix} r(R\Phi_E \mathcal{F}) \\ d(R\Phi_E \mathcal{F}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} r(\mathcal{F}) \\ d(\mathcal{F}) \end{pmatrix}.$$

② $\mu \in \mathbb{Q}$

$$\mathrm{Vect}_\mu(E) \longleftrightarrow \mathrm{Coh}_{\mathrm{tors}}(E)$$

$\square$

On higher genus curve $C/_k$ Consider the functor

$$\mathcal{M}_C(r,d) : Sch/_k \longrightarrow Set^{op}$$
$$T \longmapsto \left\{ \begin{array}{l} \text{v.b. } \mathcal{F} \text{ on } C \times T, \text{ s.t. } \forall s \in S \\ \mathcal{F}|_{C \times \{s\}} : \text{Semistable, rank} = r \\ \qquad\qquad\qquad\quad \deg = d \end{array} \right\} \Big/ \sim_{S\text{-equivalence}}$$

Thm $\exists$ Coarse moduli space $M_C(r,d)$. for the functor $\mathcal{M}_C(r,d)$.

$$\{\text{closed pt. of } M_C(r,d)\} \longleftrightarrow \left\{ \begin{array}{l} S\text{-equivalence class of rank } r \\ \text{and } \deg = d \text{ v.b. on } C \end{array} \right\}.$$

$\mathrm{Coh}(X) \xrightarrow{\text{HN}}$ Semi-Stable $\xrightarrow{\text{Jordan-Hölder}}$

$\mathcal{E}$: semi-stable. $\quad 0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_k = \mathcal{E}$. $\quad \cdots$

$\left\{ \mathcal{E}_i / \mathcal{E}_{i-1} \right\}$: stable.

$\mathcal{E} \sim \mathcal{E}'$: S-equiv. if. $\bigoplus_i \mathcal{E}_i/\mathcal{E}_{i-1} \cong \bigoplus_j \mathcal{E}'_j/\mathcal{E}'_{j-1}$.

C : $\quad \underline{\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = H^1(C, \mathcal{O}_C)} \cdot g$.

$0 \to \underline{\mathcal{O}_C} \to \mathcal{E} \to \underline{\mathcal{O}_C} \to 0 \qquad [\mathcal{E}] = [\mathcal{O}_C \oplus \mathcal{O}_C]$.

$\downarrow$

Semi-stable.

fix. rank $\mathcal{F}$, deg $\mathcal{F}$.

$h^0(\mathcal{F}(m)) = \chi(\mathcal{F}(m))$

$\hookrightarrow$ numerically

$\mathcal{F} \xrightarrow{\text{SS.}} \mathcal{F}\underline{\underline{(m)}}$: global generated. $\qquad \underline{H^0(\mathcal{F}\underline{\underline{(m)}})} \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}(m)$

CM reg. $\qquad\qquad \downarrow$

fixed CM reg

Fact $\quad$ Fix $N \in \mathbb{N}_+$, $P \in \mathbb{Q}[T]$

$$\left\{ \mathcal{F} \;\middle|\; \begin{array}{l} P_{\mathcal{F}} = P. \\ \mathcal{O}_C^{\oplus N} \twoheadrightarrow \mathcal{F} \end{array} \right\} : \underline{\text{scheme of fin. type}}.$$

Hilb. $\cdots$

Sch	Deligne-Mumford	Artin / alg
$\circ$	finite	reductive
fine	Coarse	good moduli.

GIT $\updownarrow$ geometric quotient $\qquad$ GIT $\updownarrow$ good quotient.

$\pi : X \to X/G =: Y$

$\pi_* \mathcal{O}_X = \mathcal{O}_Y$.

$M_C^{ss}(r,d)$

open $\cup$

$M_C^{s}(r,d) = \{\text{stable object}\}/\sim_{\text{isom.}}$ fine moduli.

<u>Fact</u> $(r,d)=1$. Semistable $\Leftrightarrow$ stable.

$M_C(r,d)$: fine moduli space.

<u>dim 2</u> (r, c_1, c_2). <u>Coarse moduli space</u>. dim, irreducibility, rationality.

<u>Mukai vector</u>

<u>elliptic surface</u>. $\pi : S \to C$ elliptic fibration.

<u>dim n</u> <u>$(r; c_1, c_2, \cdots, c_n)$</u>. boundedness.

char $= 0$. ✓

$\boxed{\text{char} > 0. \text{ Langer. 04. boundedness.}}$

<u>II.</u> <u>Thm</u> (Mukai 78, Mehta/Nori 84). $\square$ A : ab. var. $\mathcal{E}$: v.b. on A.

$\mathcal{E}$: semi-homogeneous $\Leftrightarrow$ $\mathcal{E}$: semistable + zero proj. Chern class.

$c(\mathcal{E}) = \left(1 + \frac{c_1(\mathcal{E})}{r}\right)^r$

<u>Pf</u> "$\Rightarrow$" Mukai 78. $\pi_r^* \mathcal{E} = \mathcal{H} \otimes (\det \mathcal{E})^{\otimes r}$

$r = \text{rk}(\mathcal{E})$.

$\mathcal{H}$: homogeneous. $\pi_r : A \to A$, $x \mapsto rx$.

" $\Leftarrow$ " ? mod. p reduction.