CHERN INEQUALITIES

BOWEN LIU

ABSTRACT. It's a lecture note for studying the paper [Miy87].

CONTENTS

0. Conventions

(1) An (algebraic) variety over a field k is an integral seperated scheme of finite type over k.
(2) A subvariety of a variety is a closed subscheme which is a variety.
(3) A curve, surface or a threefold means a variety of dimension $1,2$ or 3.
(4) A point on a scheme will always be a closed point.

1. Preliminaries

In this section, unless otherwise specified, X always denotes a variety of dimension n over an algebraically closed field k.

1.1. **Torsion-freeness and relexivity.**

1.1.1. *Torsion-freeness.*

Definition 1.1.1. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be **locally free sheaf** if there is an open covering $\{U_i\}$ of X such that $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus r}$ holds for every U_i.

Definition 1.1.2. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be **coherent sheaf** if

(1) $\mathscr{F}$ is of finite type.
(2) For every open subset $U \subseteq X$ and every morphism $\alpha\colon \mathscr{O}_U^r \to \mathscr{F}|_U$, the kernel of α is of finite type.

Definition 1.1.3. A coherent sheaf $\mathscr{F}$ on X is **torsion-free** if a stalk $\mathscr{F}_x$ is a torsion-free $\mathscr{O}_{X,x}$-module for every $x \in X$.

Definition 1.1.4. A coherent subsheaf $\mathscr{F}$ of a torsion-free sheaf $\mathscr{E}$ is said to be **saturated** if the quotient $\mathscr{E}/\mathscr{F}$ is again torsion-free.

Proposition 1.1.1. Let X, Y be two varieties and $f : X \to Y$ be a dominant morphism. Then for any torsion-free $\mathscr{O}_X$-module $\mathscr{F}$, the direct image $f_*\mathscr{F}$ is a torsion-free $\mathscr{O}_Y$-module.

Proof. See Proposition 8.4.5 in [GD71]. □

Proposition 1.1.2. Let X be a normal variety. Then every torsion-free sheaf is locally free outside a set of codimension two.

Proof. See Proposition 5.1.7 in [Ish14]. □

Corollary 1.1.1. Every torsion-free sheaf on a smooth curve is locally free.

1.1.2. *Reflexivity.*

Definition 1.1.5. A coherent $\mathscr{O}_X$-module $\mathscr{F}$ is said to be **reflexive** if the canonical homomorphism $\mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism.

Proposition 1.1.3. Every locally free sheaf is reflexive, and every reflexive sheaf is torsion-free.

Proof. It follows from the definitions. □

Proposition 1.1.4. The dual sheaf of any coherent sheaf is reflexive.

Proof. See Proposition 5.5.18 in [Kob87]. □

1.2. Chow ring.

1.2.1. *Cycles.*

Definition 1.2.1. A k**-cycle** on X is a $\mathbb{Z}$-linear combination of irreducible subvarieties of dimension k.

Notation 1.2.1. The group of all k-cycles on X is denoted by $Z_k(X)$.

Definition 1.2.2. A **Weil divisor** on X is an $(n-1)$-cycle.

Definition 1.2.3. A **Cartier divisor** on X is a global section of quotient sheaf $\mathscr{M}_X^*/\mathscr{O}_X^*$.

Definition 1.2.4. A k-cycle α on X is defined to be **rationally equivalent to zero** if there are finitely many $(k+1)$-dimensional irreducible subvarieties $W_i \subseteq X$ and non-zero rational functions. $f_i \in \mathbb{C}(W_i)$ such that

$$\alpha = \sum_i [\mathrm{div}_{W_i}(f_i)],$$

where $\mathrm{div}_{W_i}(f_i)$ is the divisor of the rational functions[1] f_i on W_i.

Definition 1.2.5. The group of k-cycles modulo rational equivalences is defined to be $A_k(X)$, which is said to be the k-th **Chow group**.

Example 1.2.1. $A_{n-1}(X)$ is the group of Weil divisors modulo linear equivalence.

Notation 1.2.2. The group of Cartier divisors modulo linear equivalence is denoted by $\mathrm{Pic}(X)$.

Remark 1.2.1. There is a group homomorphism from $\mathrm{Pic}(X)$ to $A_{n-1}(X)$. In general it's neither injective nor surjective, but it's injective when X is normal and an isomorphism when X is smooth.

Definition 1.2.6. The group of **cycles of codimension** k **modulo rational equivalence** is defined to be $A^k(X) := A_{n-k}(X)$.

1.2.2. *The intersection pairing.*

Theorem 1.2.1. Let X be a smooth variety. There is a unique intesection product $A^r(X) \times A^s(X) \to A^{r+s}(X)$ for each r, s satisfying the axioms listed below

(1) The intersection pairing makes makes $A^*(X)$ into a commutative associated graded ring with identity. It's called the **Chow ring** of X.

[1]Note that the subvariety W_i may fail to be normal, so this requires a more general definition of $\mathrm{div}_{W_i}(f_i)$ than the usual one.

(2) For any morphism $f: X \to Y$, $f^*: A^*(Y) \to A^*(X)$ is a ring homomorphism. If $g: Y \to Z$ is another morphism, then $f^* \circ g^* = (g \circ f)^*$.
(3) If $f: X \to Y$ is a proper morphism, $f_*: A^*(X) \to A^*(Y)$ is a homomorphism of graded groups. If $g: Y \to Z$ is another proper morphism, then $g_* \circ f_* = (g \circ f)_*$.
(4) If $f: X \to Y$ is a proper morphism, $x \in A^*(X)$ and $y \in A^*(Y)$, then
$$f_*(x \cdot f^* y) = f_*(x) \cdot y.$$
This is said to be the **projection formula**.
(5) If Y, Z are cycles on X, and if $\Delta: X \to X \times X$ is the diagonal morphism, then
$$Y.Z = \Delta^*(Y \times Z).$$
(6) If Y and Z are subvarieties of X which intersec properly (meaning that every irreducible component of $Y \cap Z$ has codimension equal to $\operatorname{codim} Y + \operatorname{codim} Z$), then
$$Y.Z = \sum i(Y, Z; W_j) W_j,$$
where the sum runs over the irreducible components W_j of $Y \cap Z$, and where the integer $i(Y, Z; W_j)$ depends only on a neighborhood of the generic point of W_j on X, which is said to be the **local intersection multiplicity** of Y and Z along W_j.
(7) If Y is a subvariety of X, and Z is an effective Cartier divisor meeting Y properly, then $Y.Z$ is just the cycle associated to the Cartier divisor $Y \cap Z$ on Y, which is defined by restricting the local equation of Z to Y.

Proof. See appendix A.1 in [Har77]. □

Remark 1.2.2. If X is not smooth, the intersection pairing also makes sense in some subtle setting. For example, for any variety (or scheme), there is always an intersection pairing
$$\operatorname{Pic}(X) \times A^k(X) \to A^{k+1}(X).$$

1.3. Chern classes.

1.3.1. *Chern classes of locally free sheaf.*

Definition 1.3.1. A locally free sheaf $\mathscr{E}$ of rank r on X has **Chern classes** $c_i(\mathscr{E}) \in A^i(X)$ for all $0 \le i \le r$, which is defined by
$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathscr{E}) \xi^{r-i} = 0$$
in $A^r(\mathbb{P}(\mathscr{E}))$, where $\xi \in A^1(\mathbb{P}(\mathscr{E}))$ be the class of the divisor corresponding to $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ and $\pi: \mathbb{P}(\mathscr{E}) \to X$ be the projection.

Definition 1.3.2. Let $\mathscr{E}$ be a locally free sheaf of rank r on X. The **total Chern class** is

$$c(\mathscr{E}) = c_0(\mathscr{E}) + \cdots + c_r(\mathscr{E}) \in A^*(X).$$

Proposition 1.3.1.

(1) $c_0(\mathscr{E}) = 1$ for any $\mathscr{E}$ and $c_1(\mathscr{O}_X) = 1$ for any X.
(2) If $f: X \to Y$ is a morphism and $\mathscr{E}$ is locally free on Y, then $c_i(f^*\mathscr{E}) = f^*(c_i(\mathscr{E}))$.
(3) If $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ is an exact sequence, then $c(\mathscr{F}) = c(\mathscr{E})c(\mathscr{G})$.
(4) $c_i(\mathscr{E}^\vee) = (-1)^i c_i(\mathscr{E})$, where $\mathscr{E}^\vee$ is the dual of $\mathscr{E}$.
(5) $c_1(\bigwedge^r \mathscr{E}) = c_1(\mathscr{E})$ when $\mathscr{E}$ has rank r.
(6) If D is a Cartier divisor on X, then

$$c_1(\mathscr{O}_X(D)) = D.$$

Proof. See appendix A.3 in [Har77]. □

1.3.2. *Chern classes of coherent sheaf.* Let $F(X)$ be the free abelian group generated by the set of coherent sheaves (up to isomorphisms, otherwise it's not a set) on X, that is, an element of $F(X)$ is a formal linear combination $\sum_i n_i \mathscr{F}_i$, where $n_i \in \mathbb{Z}$ and $\mathscr{F}_i$ is coherent. Let

$$(E) \quad 0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

be an exact sequence of sheaves, and we associate the element $Q(E) = \mathscr{F} - \mathscr{F}' - \mathscr{F}''$ of $F(X)$ to this exact sequence.

Definition 1.3.3. The **group of classes of sheaves** $K(X)$ on X is defined to be the quotient of $F(X)$ by the subgroup generated by the $Q(E)$, where E runs over all short exact sequences.

Definition 1.3.4. Let $F_1(X)$ be the free group generated by the set of locally free sheaves (up to isomorphisms), and $K_1(X)$ be the quotient of $F_1(X)$ by the subgroup generated by the $Q(E)$, where E runs over all short exact sequences of locally free sheaves.

Theorem 1.3.1 ([BS58])**.** Let X be a smooth quasi-projective variety. Then the homomorphism $\epsilon: K_1(X) \to K(X)$ is a bijection.

Corollary 1.3.1. The definition of Chern classes can be extended to arbitrary coherent sheaves.

1.4. **Cones of divisors and curves.**

1.4.1. *The cones of divisors.*

Definition 1.4.1. For two Cartier divisors D_1, D_2 on X, D_1 is **numerically equivalent** to D_2 if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves C.

Definition 1.4.2. The **Néron-Severi group** $N^1(X)$ is the quotient group of Cartier divisors by numerically equivalence, and

$$N^1(X)_{\mathbb{Q}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Theorem 1.4.1. The Néron-Severi group $N^1(X)$ is a free abelian group of finit rank, and the rank of $N^1(X)$ is said to be the **Picard number**.

Definition 1.4.3. For two 1-cycles C, C' on X, C is **numerically equivalent** to C' if they have the same intersection number with every Cartier divisor.

Notation 1.4.1. The quotient group of $Z_1(X)$ by numerically equivalence is denoted by $N_1(X)$, and

$$N_1(X)_{\mathbb{Q}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N_1(X)_{\mathbb{R}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Remark 1.4.1. The intersection pairing

$$N^1(X) \times N_1(X) \to \mathbb{Z}$$

is by definition non-degenerate.

Definition 1.4.4. The **cone of effective curves** $\mathrm{NE}(X)_{\mathbb{R}} \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by non-negative linear combinations of curves, and $\overline{\mathrm{NE}}(X)_{\mathbb{R}}$ is the **cone of pseudo-effective curves**, where $N_1(X)_{\mathbb{R}}$ is endowed with its usual topology as a $\mathbb{R}$-vector space.

1.4.2. *Nef cones and ample cones.*

Definition 1.4.5. A Cartier divisor on X is **nef (numerically effective)** if it has non-negative intersection with every irreducible curve on X.

Definition 1.4.6. The ample classes in $N^1(X)_{\mathbb{R}}$ forms an open cone $\mathrm{NA}(X)_{\mathbb{R}}$, which is said to be **ample cone**.

Definition 1.4.7. The nef classes in $N^1(X)_{\mathbb{R}}$ forms a closed cone $\mathrm{Nef}(X)_{\mathbb{R}}$, which is said to be **nef cone**.

Theorem 1.4.2. Let X be a projective variety.

(1) The closure of the ample cone is the nef cone;
(2) The interior of nef cone is the ample cone.

Proof. See Theorem 1.4.23 in [Laz04]. □

Theorem 1.4.3. Let X be a projective variety.

(1) The pseudo-effective cone is the closed cone dual to the nef cone, that is,
$$\overline{\mathrm{NE}}(X)_{\mathbb{R}} = \{\gamma \in N_1(X)_{\mathbb{R}} \mid D\cdot\gamma \geq 0, \quad \forall\ D \in \overline{\mathrm{NA}}(X)_{\mathbb{R}}\}.$$

(2)
$$\mathrm{NA}(X)_{\mathbb{R}} = \{\gamma \in N^1(X)_{\mathbb{R}} \mid D\cdot\gamma > 0, \quad \forall\ D \in \overline{\mathrm{NE}}(X)_{\mathbb{R}} - \{0\}\}.$$

Proof. See Theorem 1.4.28 and Theorem 1.4.29 in [Laz04]. □

1.5. **Asymptotic Riemann-Roch.**

Theorem 1.5.1. Let X be a projective variety of dimension n and D be a Cartier divisor on X. Then
$$\chi(X,\mathscr{O}(mD)) = \frac{D^n}{n!}m^n + O(m^{n-1}).$$
More generally, for any coherent sheaf $\mathscr{F}$ on X,
$$\chi(X,\mathscr{F}\otimes\mathscr{O}_X(mD)) = \operatorname{rank}\mathscr{F}\cdot\frac{D^n}{n!}m^n + O(m^{n-1}).$$

Proof. See Theorem 1.1.24 in [Laz04]. □

2. TECHNIQUES

2.1. **Semistable sheaves.** Let X be a normal projective variety of dimension n over an algebraically closed field k of arbitrary characteristic.

Definition 2.1.1. The **average first Chern class** of a torsion-free sheaf $\mathscr{E}$ is

$$\delta(\mathscr{E}) = \frac{c_1(\mathscr{E})}{\operatorname{rank}\mathscr{E}} \in A^1(X)_{\mathbb{Q}}.$$

Definition 2.1.2. For a given $(n-1)$-tuple $\mathfrak{A} = (H_1, \dots, H_{n-1}) \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$, the **average degree (or slope)** with respect to $\mathfrak{A}$ is the rational number $\delta_{\mathfrak{A}}(\mathscr{E}) = \delta(\mathscr{E})H_1 \dots H_{n-1}$.

Definition 2.1.3. A torsion-free sheaf $\mathscr{E}$ is said to be **semistable** if

$$\delta_{\mathfrak{A}}(\mathscr{F}) \leq \delta_{\mathfrak{A}}(\mathscr{E})$$

for every non-zero subsheaf $\mathscr{F}$ of $\mathscr{E}$.

Notation 2.1. If $\mathfrak{A} = ([H], \dots, [H])$, we use the terminology H-semistable instead of $\mathfrak{A}$-semistable.

Theorem 2.1.1 ([HN75])**.** Let $\mathscr{E}$ be a torsion-free sheaf on X and $\mathfrak{A} \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$. Then there exists a unique filtration $\Sigma_{\mathfrak{A}}$,

$$0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_s = \mathscr{E},$$

which is called the **Harder-Narasimhan filtration**, such that

(1) $\mathrm{Gr}_i(\Sigma_{\mathfrak{A}}) = \mathscr{E}_i/\mathscr{E}_{i+1}$ is a torsion-free $\mathfrak{A}$-semistable sheaf;
(2) $\delta_{\mathfrak{A}}(\mathrm{Gr}_i(\Sigma_{\mathfrak{A}}))$ is a strictly decreasing function in i.

Sketch. Here we only give a sketch of proof of the existence. Put $\delta_{\mathfrak{A}}^{\max}(\mathscr{E}) := \sup\{\delta_{\mathfrak{A}}(\mathscr{F}) \mid 0 \neq \mathscr{F} \subseteq \mathscr{E} \text{ a coherent subsheaf}\}$. Then we need to prove that

(1) $\delta_{\mathfrak{A}}^{\max}(\mathscr{E}) < \infty$;
(2) There exists a saturated subsheaf $\mathscr{F}_1 \subseteq \mathscr{E}$ with maximal slope.

After that, suppose both $\mathscr{F}_1$ and $\mathscr{F}_2$ coherent subsheaves of rank r_1 and r_2 with maximal slope. By the following exact sequence

$$0 \to \mathscr{F}_1 \cap \mathscr{F}_2 \to \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 + \mathscr{F}_2 \to 0,$$

one has

$$c_1(\mathscr{F}_1 + \mathscr{F}_2) = c_1(\mathscr{F}_1) + c_1(\mathscr{F}_2) - c_1(\mathscr{F}_1 \cap \mathscr{F}_2)$$
$$\operatorname{rank}(\mathscr{F}_1 + \mathscr{F}_2) = \operatorname{rank}(\mathscr{F}_1) + \operatorname{rank}(\mathscr{F}_2) - \operatorname{rank}(\mathscr{F}_1 \cap \mathscr{F}_2).$$

Then

$$\begin{aligned}\operatorname{rank}(\mathscr{F}_1+\mathscr{F}_2)\delta_{\mathfrak{A}}(\mathscr{F}_1+\mathscr{F}_2) &= r_1\delta_{\mathfrak{A}}(\mathscr{F}_1)+r_2\delta_{\mathfrak{A}}(\mathscr{F}_2)-\operatorname{rank}(\mathscr{F}_1\cap\mathscr{F}_2)\delta_{\mathfrak{A}}(\mathscr{F}_1\cap\mathscr{F}_2)\\ &\geq (r_1+r_2)\delta_{\mathfrak{A}}^{\max}(\mathscr{E})-\operatorname{rank}(\mathscr{F}_1\cap\mathscr{F}_2)\delta_{\mathfrak{A}}^{\max}(\mathscr{E})\\ &= \operatorname{rank}(\mathscr{F}_1+\mathscr{F}_2)\delta_{\mathfrak{A}}^{\max}(\mathscr{E}).\end{aligned}$$

This shows $\mathscr{F}_1+\mathscr{F}_2$ also has maximal slope. By adding all these subsheaves together, this gives the **maximal $\mathfrak{A}$-destabilizing subsheaf** $\mathscr{E}_1$. We repeat above process to obtain the maximal $\mathfrak{A}$-destabilizing subsheaf of $\mathscr{E}/\mathscr{E}_1$, and consider its preimage to obtain $\mathscr{E}_2$, that is, $\mathscr{E}_2/\mathscr{E}_1=(\mathscr{E}/\mathscr{E}_1)_1$. It remains to show $\delta_{\mathfrak{A}}(\mathscr{E}_1)>\delta_{\mathfrak{A}}(\mathscr{E}_2/\mathscr{E}_1)$. Indeed, otherwise we would have $\delta_{\mathfrak{A}}(\mathscr{E}_1)\leq\delta_{\mathfrak{A}}(\mathscr{E}_2)$, a contradiction. □

Remark 2.1.1. The maximal $\mathfrak{A}$-destabilizing subsheaf of $\mathscr{E}$ is characterized by the following properties:

(1) $\delta_{\mathfrak{A}}(\mathscr{E}_1)\geq\delta_{\mathfrak{A}}(\mathscr{F})$ for every coherent subsheaf $\mathscr{F}$ of $\mathscr{E}$;
(2) If $\delta_{\mathfrak{A}}(\mathscr{E}_1)=\delta_{\mathfrak{A}}(\mathscr{F})$ for $\mathscr{F}\subset\mathscr{E}$, then $\mathscr{F}\subset\mathscr{E}_1$.

Remark 2.1.2. The $\mathfrak{A}$-semistable filtration of the dual sheaf $\mathscr{E}^*$ is essentially the same as that of $\mathscr{E}$, with each entry substituted by the duals of the quotient $\mathscr{E}/\mathscr{E}_{s-i}$.

Theorem 2.1.2. Let $\mathscr{E}_1^{\mathfrak{A}}\subset\mathscr{E}$ denote the maximal $\mathfrak{A}$-destabilizing subsheaf for $\mathfrak{A}\in\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$.

(1) Let L be a closed affine segment joining $\mathfrak{A},\mathfrak{C}\in\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$ and $\mathfrak{B}=(1-t)\mathfrak{A}+t\mathfrak{C}$ be a rational point on L. Then $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}})=\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ whenever $0<t<\epsilon$, where ϵ is a positive constant depends continously on $\mathfrak{C}$ provided $\mathscr{E}$ and $\mathfrak{A}$ is fixed.
(2) Let $K\subset\overline{\mathrm{NA}}(X)_{Q}^{n-1}$ be a compact subset and $\mathfrak{A}\in\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$ is away from K. Let $\mathfrak{A}\sharp K$ stands the union of the segments joining $\mathfrak{A}$ and K. Then there exists an open neighborhood $U\subset N^1(X)_{\mathbb{Q}}$ of $\mathfrak{A}$ such that $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}})=\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B}\in U\cap(\mathfrak{A}\sharp K)\cap\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$.
(3) If $\mathfrak{A}\in\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$, then there exists an open neighborhood $U\subset\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$ of $\mathfrak{A}$ such that $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}})=\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B}\in U$.

Proof. For simplicity, we show the case $n=2$ only, and the proof is quite similar for higher dimensions.

(1). Suppose $\mathfrak{C}=H\in\overline{\mathrm{NA}}(X)_{\mathbb{Q}}$. If $\mathscr{E}^*(H)$ is globally generated, that is, there exists a surjective morphism $\mathscr{O}_X^{\oplus N}\to\mathscr{E}^*(H)$ for some integer N. By taking dual we have an injective morphism $\mathscr{E}\to\mathscr{O}_X^{\oplus N}(H)$, and thus

$$\delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{B}})\leq c,$$

where c is a constant depending on $\mathscr{E}$, and on $\mathfrak{C}$ continously. If H is ample, then there exists some integer m such that mH is globally generated, and thus in this case $\delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{B}}) \le c$ for some constant c depending on $\mathscr{E}$, and on $\mathfrak{C}$ continously. Finally if $H \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}$, we also have the same result, as it's a limit of ample divisors. Furthermore, we put $c' = \delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{A}})$. By the definition of the maximal destabilizing sheaves, we get

$$\delta_{\mathfrak{B}}(\mathscr{E}_1^{\mathfrak{A}}) \le \delta_{\mathfrak{B}}(\mathscr{E}_1^{\mathfrak{B}}).$$

As $\delta_{\mathfrak{B}}$ is a linear function in $\mathfrak{B} = (1-t)\mathfrak{A} + t\mathfrak{C}$, this inequality is rewritten as

$$(1-t)\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}}) + t\delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{A}}) \le (1-t)\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}}) + t\delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{B}}).$$

Hence

$$\begin{aligned}\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}}) \le \delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}}) &\le \delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}}) + \frac{t}{1-t}(\delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{B}}) - \delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{A}}))\\ &\le \delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}}) + \frac{t}{1-t}(c-c').\end{aligned}$$

Note that $\delta(\mathscr{E}_1^{\mathfrak{A}}), \delta(\mathscr{E}_1^{\mathfrak{B}}) \in (1/r!)A^1(X)_{\mathbb{Z}}$ and $\mathfrak{A} \in (1/m)N^1(X)_{\mathbb{Z}}$ for some positive integer m. Therefore, if

$$\frac{t}{1-t}(c-c') < \frac{1}{r!m},$$

then $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}}) = \delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}})$.

(2). Let U be the open ball centered at $\mathfrak{A}$ with radius r, where $r = \inf_{\mathfrak{C}\in K} \epsilon(\mathscr{E}, \mathfrak{A}, \mathfrak{C}) d(\mathfrak{A}, \mathfrak{C})$, d standing for Euclidean metric.

(3). Let $K \subset \mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$ be a sphere centered at $\mathfrak{A}$ and apply (2). □

Corollary 2.1.1. Given a compact subset $K \subset \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$ and $\mathfrak{A} \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$ is away from K, the $\mathfrak{B}$-semistable filtration is a refinement of $\mathfrak{A}$-semistable filtration for all $\mathfrak{B} \in \mathfrak{A}\sharp K$ sufficiently near $\mathfrak{A}$.

Proof. By (2) of above theorem, we have $\mathscr{E}_1^{\mathfrak{B}} \subseteq \mathscr{E}_1^{\mathfrak{A}}$ for all $\mathfrak{B} \in \mathfrak{A}\sharp K$ sufficiently near $\mathfrak{A}$. If $\mathscr{E}$ is semistable, it's clear that the $\mathfrak{B}$-semistable filtration of $\mathscr{E}$ is a refinement of $\mathfrak{A}$-semistable filtration of $\mathscr{E}$, and the general case is obtained by repeating above process for each semistable grade $\mathscr{E}_i/\mathscr{E}_{i+1}$. □

Corollary 2.1.2. Let $\mathscr{E}$ be a torsion-free sheaf on X. Then the function $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ is continous on $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$, and is continous on any rational segment of $\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$.

Proof. Note that if both $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$, then there exists some open neighborhood of $\mathfrak{A}$ containing $\mathfrak{B}$, and there also exists some open neighborhood of $\mathfrak{B}$ containing $\mathfrak{A}$. By the symmetry we have $\mathscr{E}_1^{\mathfrak{B}} = \mathscr{E}_1^{\mathfrak{A}}$, and thus

$\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ is continous on $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$. The same arguement shows $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ is also continous in any rational segment of $\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$. □

2.2. **A numerical criterion for semistability on curves.** Throught this section, the ground field k is always an algebraically closed field with characteristic 0 except Lemma 2.2.1, and C is a smooth complete curve.

2.2.1. *Projective bundle on curves.* Let $\mathscr{E}$ be a locally free sheaf of rank r on C and $\pi\colon \mathbb{P}(\mathscr{E}) \to C$ the associated projective bundle with tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

Definition 2.2.1. The **normalized hyperplane class** $\lambda_{\mathscr{E}}$ is the numerical class of $c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) - \pi^*\delta(\mathscr{E}) \in N^1(\mathbb{P}(\mathscr{E}))_{\mathbb{Q}}$.

Remark 2.2.1. The normalized hyperplane class $\lambda_{\mathscr{E}}$ is uniquely determined by two properties:

(1) $\lambda_{\mathscr{E}}^r = 0$.
(2) $\lambda_{\mathscr{E}}$ on each fiber is numerically equivalent to the hyperplane.

Proposition 2.2.1. The class of relative anti-canonical divisor $-K_{\mathbb{P}(\mathscr{E})} + \pi^*K_C$ equals $r\lambda_{\mathscr{E}}$.

Proof. It follows from the relative Euler sequence, that is,

$$0 \to \Omega^1_{\mathbb{P}(\mathscr{E})/C} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1) \otimes \pi^*\mathscr{E}^* \to\to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to 0.$$

□

Proposition 2.2.2.

(1) $\pi_*(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(\ell)) = \mathscr{S}^\ell\mathscr{E}$ for $\ell \geq 0$ and $\pi_*(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(\ell)) = 0$ for $\ell < 0$.
(2) $R^i\pi_*(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(\ell)) = 0$ for $0 < i < n$.
(3) $R^n\pi_*(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(\ell)) = 0$ for $\ell > -n-1$.

Proof. See Exercise III 8.4 in [Har77]. □

Proposition 2.2.3. The Néron-Severi group of $\mathbb{P}(\mathscr{E})$ is

$$N^1(\mathbb{P}(\mathscr{E})) = \mathbb{R}\lambda_{\mathscr{E}} \oplus \pi^*N^1(X),$$

and the group of numerically equivalent 1-cycles is

$$N_1(\mathbb{P}(\mathscr{E})) = \lambda_{\mathscr{E}}^{r-2}N^1(\mathbb{P}(\mathscr{E})).$$

Proof. See Proposition V 2.3 in [Har77]. □

2.2.2. *Criterion.*

Lemma 2.2.1. Let f be a seperable surjective k-morphism of a smooth complete curve C' onto C. Then a locally free sheaf $\mathscr{E}$ is semistable if and only if $f^*\mathscr{E}$ is semistable.

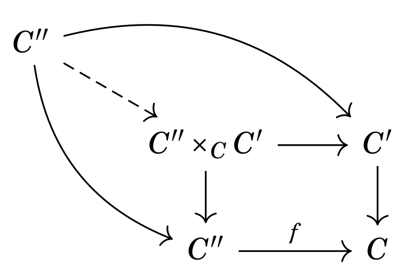
Proof. Firstly let's prove "if" part. Let $\mathscr{G} \subseteq \mathscr{E}$ be a non-zero subsheaf. Then $\delta(f^*\mathscr{G}) \le \delta(f^*\mathscr{E})$ as $f^*\mathscr{E}$ is semistable, and thus $\delta(\mathscr{G}) \le \delta(\mathscr{F})$.

Conversely, suppose $\mathscr{E}$ is semistable. Without lose of generality we may assume f is a Galois morphism with Galois group G, which acts on $f^*\mathscr{E}$. If $f^*\mathscr{E}$ is not semistable and $\mathscr{F}_1$ be the maximal destabilizing subbundle of $f^*\mathscr{E}$. For any $g \in G$, we have $g^*\mathscr{F}_1 = \mathscr{F}_1$ as the maximal destabilizing subsheaf is unique. Hence there exists a subbundle $\mathscr{E}_1$ of $\mathscr{E}$ such that $f^*\mathscr{E}_1 = \mathscr{F}_1$, and by "if" part $\mathscr{E}_1$ is semistable. On the other hand, by semistability we have $\mathscr{E}_1 = \mathscr{E}$, and thus $\mathscr{F}_1 = f^*\mathscr{E}$. This completes the proof. □

Theorem 2.2.1. The following conditions are equivalent:

(1) $\mathscr{E}$ is semistable;
(2) $\lambda_{\mathscr{E}}$ is nef;
(3) $\overline{\mathrm{NA}}(\mathbb{P}(\mathscr{E})) = \mathbb{R}_+\lambda_{\mathscr{E}} \oplus \mathbb{R}_+\pi^*d$, where d is a positive generator of $N^1(C)_{\mathbb{Z}} \cong \mathbb{Z}$;
(4) $\overline{\mathrm{NE}}(\mathbb{P}(\mathscr{E})) = \mathbb{R}_+\lambda_{\mathscr{E}}^{r-1} \oplus \mathbb{R}_+\lambda_{\mathscr{E}}^{r-2}\pi^*d$;
(5) Every effective divisor on $\mathbb{P}(\mathscr{E})$ is nef.

Proof. (1) to (2). If $\lambda_{\mathscr{E}}$ is not nef, then there exists an irreducible curve $C' \subset \mathbb{P}(\mathscr{E})$ with $C'\lambda_{\mathscr{E}} < 0$. It's clear[2] that C' is mapped surjectively onto C. Let C'' be the normalization of C' and $f: C'' \to C$ be the composition of $C'' \to C' \to C$. Then by the base change $f: C'' \to C$, the multi-section C' becomes a union of cross sections C''_i on the projective bundle $\mathbb{P}(f^*\mathscr{E})$ over C'', and $C''_i\lambda_{\mathbb{P}(f^*\mathscr{E})}$ is evidently negative since $C'\lambda_{\mathscr{E}} < 0$. For a section $s: C \to C''_i \subset \mathbb{P}(f^*\mathscr{E})$, it gives a line bundle $\mathscr{L} = s^*\mathscr{O}_{\mathbb{P}(f^*\mathscr{E})}(1)$ on C, which has degree $C''_i c_1(\mathscr{O}_{\mathbb{P}(f^*\mathscr{E})}(1)) = C''_i\lambda_{f^*\mathscr{E}} + \delta(f^*\mathscr{E}) < \delta(f^*\mathscr{E})$, so that $f^*\mathscr{E}$ is unstable, and thus $\mathscr{E}$ is unstable by Lemma 2.2.1.



[2]Otherwise we have $C'\lambda_{\mathscr{E}} > 0$.

(2) to (4). If $\lambda_{\mathscr{E}}^{r-2}(a\lambda_{\mathscr{E}} + b\pi^* d)$ is pseudo-effective and $\lambda_{\mathscr{E}}$ is nef, then

$$b = \lambda_{\mathscr{E}}^{r-1}(a\lambda_{\mathscr{E}} + b\pi^* d) \geq 0.$$

On the other hand, $\lambda_{\mathscr{E}}^{r-1}$ is pseudo-effective since $\lambda_{\mathscr{E}}$ is nef, and thus $a \geq 0$.

The equivalent between (3) and (4) is straightforward since the nef cone is the closed cone dual to the pseudo-effective cone (Theorem 1.4.3).

(3) and (4) to (5). Since $\lambda_{\mathscr{E}}$ is nef, $\lambda_{\mathscr{E}} + \epsilon\pi^* d$ is ample for any positive real number ϵ. Assume $a\lambda_{\mathscr{E}} + b\pi^* d$ is an effective divisor. Then the 1-cycles $(a\lambda_{\mathscr{E}} + b\pi^* d)(\lambda_{\mathscr{E}} + \epsilon\pi^* d)^{r-2}$ is effective, and thus their limit $(a\lambda_{\mathscr{E}} + b\pi^* d)\lambda_{\mathscr{E}}^{r-2}$ is pseudo-effective. Then by (4) one has $a, b \geq 0$, and thus $a\lambda_{\mathscr{E}} + b\pi^* d$ is nef by (3).

(5) to (1). Suppose that $\mathscr{E}$ is unstable and let $\mathscr{E}_1$ be the maximal destabilizing subbundle. Let α be a rational number with $\delta(\mathscr{E}_1) > \alpha > \delta(\mathscr{E})$. Then by the Riemann-Roch theorem,

$$\begin{aligned} H^0(C, \mathscr{S}^N \mathscr{E}_1(-N\alpha d)) &\subseteq H^0(C, \mathscr{S}^N \mathscr{E}(-N\alpha d)) \\ &\cong H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(N) \otimes \pi^* \mathscr{O}_C(-N\alpha d))) \end{aligned}$$

is non-trivial for sufficiently large N. Then $N\{\lambda_{\mathscr{E}} + (\delta(\mathscr{E}) - \alpha)\pi^* d\}$ is effective but clearly not nef. □

2.2.3. *Semipositive and semistability.*

Definition 2.2.2. Let D be a $\mathbb{Q}$-Cartier divisor on C. A $\mathbb{Q}$-torsion-free sheaf $\mathscr{F} = \mathscr{E}(D)$ is said to be **ample** or **semipositive** if $\xi + \pi^* D$ is ample or nef, where $\xi = c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$.

Definition 2.2.3. A $\mathbb{Q}$-torsion-free sheaf $\mathscr{F}$ is said to be **negative** or **seminegative** if $\mathscr{F}^*$ is ample or semipositive.

Proposition 2.2.4. The direct sums, tensor products, symmetric products and exterior products of ample (or semipositive) $\mathbb{Q}$-torsion-free sheaves are all ample (or semipositive).

Theorem 2.2.2. Let $\mathscr{E}$ be a vector bundle on C. Then $\mathscr{E}$ is semistable if and only if $\mathscr{E}(-\delta(E))$ is semipositive.

Proof. It follows from Theorem 2.2.1. □

Corollary 2.2.1. Let $\mathscr{E}$ be a vector bundle on C. Then $\mathscr{E}$ is semistable if and only if $\mathscr{E}(-\delta(E))$ is seminegative.

Proof. It suffices to note that $\mathscr{E}$ is semistable if and only if $\mathscr{E}^*$ is semistable. □

Corollary 2.2.2.

(1) The $\mathbb{Q}$-vector bundle $\mathscr{E}(-D)$ is seminegative if and only if $\deg D \geq \deg\delta(\mathscr{E}_1)$, where $\mathscr{E}_1$ is the maximal destabilizing subsheaf of $\mathscr{E}$.
(2) The $\mathbb{Q}$-vector bundle $\mathscr{E}(-D)$ is negative if and only if $\deg D > \deg\delta(\mathscr{E}_1)$, where $\mathscr{E}_1$ is the maximal destabilizing subsheaf of $\mathscr{E}$.
(3) The $\mathbb{Q}$-vector bundle $\mathscr{E}(D)$ is semipositive if and only if $\deg D \geq \deg\delta((\mathscr{E}^*)_1)$.
(4) The $\mathbb{Q}$-vector bundle $\mathscr{E}(D)$ is positive if and only if $\deg D > \deg\delta((\mathscr{E}^*)_1)$.

Proof. For simplicity we only prove the first statement, and the proof is quite similar for others.

Let $\mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \mathscr{E}$ be the semistable filtration of $\mathscr{E}$. Since $\mathscr{G}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ is semistable and $\deg\delta(\mathscr{G}_i)$ is decreasing in i, one has $\mathscr{G}_i(-\delta(\mathscr{E}_1))$ is seminegative for all i, and thus $\mathscr{E}(-\delta(\mathscr{E}_1))$ is seminegative. If $\deg D \geq \deg\delta(\mathscr{E}_1)$, then $\mathscr{E}(-D)$ is also seminegative.

Conversely, if $\deg D$ is smaller than $\deg\delta(\mathscr{E}_1)$ for a $\mathbb{Q}$-divisor D, then $\mathscr{E}(-D)$, containing an ample $\mathbb{Q}$-vector bundle $\mathscr{E}_1(-D)$, is never seminegative. □

Corollary 2.2.3. A semistable vector bundle $\mathscr{E}$ on C is ample (resp. semipositive, seminegative, negative) if and only if its degree is positive (resp. semipositive, seminegative, negative).

Proof. Take $D = 0$ in Corollary 2.2.2. □

Corollary 2.2.4. Let $\mathscr{E}$ and $\mathscr{F}$ be semistable bundles on C. Then $\mathscr{E} \otimes \mathscr{F}$ and $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ are also semistable.

Proof. It follows from the semipositive bundle tensor with semipositive bundle is still semipositive. □

Corollary 2.2.5. Let $\mathscr{E}$ and $\mathscr{F}$ be two vector bundles. Then $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ is negative if and only if $\deg\delta(\mathscr{F}_1) + \deg\delta((\mathscr{E}^*)_1) < 0$. As a consequence, $\mathscr{H}om(\mathscr{E}_1, \mathscr{E}/\mathscr{E}_1)$ is negative.

Proof. For the first part, note that $\mathscr{H}om(\mathscr{E}, \mathscr{F}) = \mathscr{E}^* \otimes \mathscr{F}$ and take $D = 0$ in Corollary 2.2.2. For the half part, it suffices to note $(\mathscr{E}/\mathscr{E}_1)_1 = \mathscr{E}_2/\mathscr{E}_1$. □

Corollary 2.2.6. A vector bundle is semistable if and only if $\mathscr{S}^n\mathscr{E}$ is semistable, where $n \geq 2$.

Proposition 2.2.5. Let $\mathscr{E}$ be a vector bundle on C. The following conditions are equivalent:

(1) $\mathscr{E}$ is semistable;
(2) $\mathscr{E}(-D)$ is negative with D is a $\mathbb{Q}$-divisor of degree $\delta(\mathscr{E}) + (1/2r!)$.

Proof. The implication (1) to (2) follows from Corollary 2.2.1.

Conversely, assume (2) and let $\mathscr{E}_1$ be the maximal destabilizing subsheaf. Then by Corollary 2.2.2 we have $\mathscr{E}(-D)$ is negative if and only if $\deg D > \deg\delta(\mathscr{E}_1)$ so that

$$\delta(\mathscr{E}) \le \delta(\mathscr{E}_1) < \delta(\mathscr{E}) + \frac{1}{2r!}.$$

On the other hand, both $\deg\delta(\mathscr{E}_1)$ and $\deg\delta(\mathscr{E})$ sit in $(1/r!)\mathbb{Z}$. Hence we have $\deg\delta(\mathscr{E}_1) = \deg\delta(\mathscr{E})$, and thus $\mathscr{E}_1 \cong \mathscr{E}$. □

Corollary 2.2.7. Let $\mathscr{C} \to T$ be a proper smooth family of irreducible curves, where $\mathscr{C}$ and T are k-varieties. Let $\mathscr{E}$ be a vector bundle on $\mathscr{C}$. Then the set

$$S(T) = \{t \in T \mid \mathscr{E} \text{ is semistable on } C_t\}$$

is a Zariski open subset of T.

2.3. **Mumford-Mehta-Ramanathan's theorem.**

Theorem 2.3.1 ([MR82])**.** Let X be a complex normal projective variety of dimension n and $\mathscr{E}$ be a torsion-free sheaf. Let $H_1,\dots,H_{n-1}$ be ample Cartier divisors. Then for sufficiently large integers $m_1,\dots,m_{n-1}$, the maximal destabilizing subsheaf $\mathscr{F}$ of $\mathscr{E}|_C$ extends to a saturated subsheaf of $\mathscr{E}$ on X if C is a general complete intersection curve of $|m_iH_i|$'s. (Such an extension of $\mathscr{F}$ is necessarily the maximal $(H_1,\dots,H_{n-1})$-destabilizing subsheaf of $\mathscr{E}$ and hence unique.)

2.4. **The Bogomolov-Gieseker inequality for semistable sheaves.** In this section, the ground field k is always algebraically closed of characteristic zero.

Lemma 2.4.1. Let X be a normal projective variety of dimension n and $\mathfrak{A} \in \mathrm{NA}(X)^{n-1}$. Let $\mathscr{E}$ be an $\mathfrak{A}$-semistable torsion-free sheaf on X, with its first Chern class being a $\mathbb{Q}$-Cartier divisor. Let D be a non-zero effective Cartier divisor on X. Then

$$H^0(X, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}) - D)) = 0$$

for every positive integer t such that $tc_1(\mathscr{E})$ is an integral Cartier divisor.

Proof. For a generic curve C in X, by Theorem 2.3.1 one has $\mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}))|_C$ is semistable since $\mathscr{S}^{rt}\mathscr{E}$ is semistable. If

$$H^0(X, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}) - D)) \neq 0,$$

then there is an inclusion $\mathscr{O}_C(D) \to \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}))|_C$. But $\deg\delta(\mathscr{O}_C(D)) > 0$ since D is effective and $\mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E})) = \mathscr{S}^{rt}\{\mathscr{E}(-\delta(\mathscr{E}))\}$ has degree zero on every curve. This contradicts to $\mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}))|_C$ is semistable. □

Corollary 2.4.1. Let things be as Lemma 2.4.1 and L be a fixed Cartier divisor. Then $h^0(X, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E})+L))$ is bounded by a polynomial of degree $r-1$ in t.

Proof. For simplicity of the notation, put $\mathscr{F}^t = \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}))$. The proof is by induction on the dimension n of X. If $n = 1$, let D be a reduced effective divisor of degree $d > \deg L$. Then there is a natural exact sequence

$$H^0(X, \mathscr{F}^t(-D)) \to H^0(X, \mathscr{F}^t(L)) \to H^0(D, \mathscr{F}^t(L))$$

of which the first term vanishes by Lemma 2.4.1, where the last term is a k-vector space of dimension $d\binom{rt+r-1}{rt} = d\binom{rt+r-1}{r-1}$. This completes the proof of $n = 1$.

For $n \geq 2$, let $\mathfrak{A} = (H_1, \ldots, H_n)$ in $\mathrm{NA}(X)^{n-1}$, where H_i is integral and ample. Let Y be a general hyperplane section in $|mH_i|$ for sufficiently large m such that $\mathscr{E}|_Y$ is $(H_1, \ldots, H_{n-2})$-semistable on Y and $Y - L$ is ample (Note that such a number m, though possible very large, is independent of t). Consider the exact sequence

$$H^0(X, \mathscr{F}^t(L-Y)) \to H^0(X, \mathscr{F}^t(L)) \to H^0(Y, \mathscr{F}^t(L)).$$

The first term vanishes by Lemma 2.4.1 and the dimension of the last term is bounded by a polynomial of degree $r-1$ by the induction hypothesis. This completes the proof. □

Theorem 2.4.1 (The Bogomolov-Gieseker inequality)**.** Let S be a smooth projective surface over k. If $\mathscr{E}$ is an H-semistable torsion-free sheaf of rank r on S, where H is an ample divisor, then

$$(r-1)c_1^2(\mathscr{E}) \leq 2rc_2(\mathscr{E}).$$

Proof. From Corollary 2.4.1, it follows that neither $h^0(S, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E})))$ nor $h^2(S, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E})) = h^0(S, \mathscr{S}^{rt}\mathscr{E}^*(-tc_1(\mathscr{E}^*))+K_S)$ grows like t^{r+1}. Hence we obtain the inequality

$$\chi(S, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}))) \leq \text{polynomial of degree } r \text{ in } t.$$

On the other hand, we have

$$\begin{aligned}\chi(S, \mathscr{S}^{rt}\mathscr{E}(-tc_1(\mathscr{E}))) &\overset{(1)}{=} \chi(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(rt) \otimes \pi^*\mathscr{O}_S(-tc_1(\mathscr{E}))) \\ &\overset{(2)}{=} \frac{t^{r+1}}{(r+1)!}\left\{rc_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) - \pi^*c_1(\mathscr{E})\right\}^{r+1} + O(t^r) \\ &\overset{(3)}{=} \frac{(rt)^{r+1}}{(r+1)!}\left\{-c_2(\mathscr{E}) + \frac{r-1}{2r}c_1^2(\mathscr{E})\right\} + O(t^r),\end{aligned}$$

where

(1) holds from the projection formula;
(2) holds from by the asymptotic Riemann-Roch theorem (Theorem 1.5.1);
(3) holds from the following standard computation

$$\begin{aligned}\left\{\xi-\frac{\pi^*c_1(\mathscr{E})}{r}\right\}^{r+1}&=\left\{\xi^r-\pi^*c_1(\mathscr{E})\xi^{r-1}+\frac{r-1}{2r}\pi^*c_1^2(\mathscr{E})\xi^{r-2}\right\}\left\{\xi-\frac{\pi^*c_1(\mathscr{E})}{r}\right\}\\&=\left\{\pi^*c_2(\mathscr{E})\xi^{r-2}+\frac{r-1}{2r}\pi^*c_1^2(\mathscr{E})\xi^{r-2}\right\}\left\{\xi-\frac{\pi^*c_1(\mathscr{E})}{r}\right\}\\&=\left\{-c_2(\mathscr{E})+\frac{r-1}{2r}c_1^2(\mathscr{E})\right\},\end{aligned}$$

where $\xi=c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$.

This completes the proof. □

Corollary 2.4.2. Let $\mathscr{E}$ be a locally free sheaf of rank r on a smooth surface S. Let L be an ample integral divisor on S such that $\mathscr{E}(-\delta(\mathscr{E})+L)$ is ample and $\mathscr{E}(-\delta(\mathscr{E})-L)$ is negative (as $\mathbb{Q}$-vector bundles). Assume the inequality $2rc_2(\mathscr{E})<(r-1)c_1^2(\mathscr{E})$ and put

$$\alpha=\frac{(r-1)c_1^2(\mathscr{E})-2rc_2(\mathscr{E})}{6r^2(r+1)L^2}\in\mathbb{Q}.$$

Then

(1) Either $\mathscr{S}^t\mathscr{E}(-t\delta(\mathscr{E}))$ or $\mathscr{S}^t\mathscr{E}^*(-t\delta(\mathscr{E}^*))$ contains the ample line bundle $\mathscr{O}_S(t\alpha L)$, where t is any very large integer such that $t\delta(\mathscr{E})$ and $t\alpha$ are integral.
(2) For any nef divisor D, the maximal D-destabilizing subsheaf $\mathscr{E}_1^D$ has normalized degree not less that

$$\delta_D(\mathscr{E})+\frac{\alpha LD}{r}$$

with respect to D.

Proof. (1). For simplicity, we put $\mathscr{F}=\mathscr{E}(-\delta(\mathscr{E}))$. By the standard computation we have

$$\chi(S,\mathscr{S}^t\mathscr{F})=\frac{1}{(r+1)!}\left\{-c_2(\mathscr{E})+\frac{r-1}{2r}c_1^2(\mathscr{E})\right\}+O(t^r).$$

Hence, by the Serre duality, we infer that $h^0(S,\mathscr{S}^t\mathscr{F})$ or $h^0(S,\mathscr{S}^t\mathscr{F}^*+K_S)$ is

$$\geq\frac{1}{4(r+1)!r}\left\{(r-1)c_1^2(\mathscr{E})-2rc_2(\mathscr{E})\right\}+O(t^r).$$

Assume the first case and consider the following natural exact sequences

$$0 \to H^0(S, \mathscr{S}^t\mathscr{F}(-t\alpha L)) \to H^0(S, \mathscr{S}^t\mathscr{F}) \to H^0(C, \mathscr{S}^t\mathscr{F}),$$
$$0 \to H^0(C, \mathscr{S}^t\mathscr{F}(-tL)) \to H^0(C, \mathscr{S}^t\mathscr{F}) \to H^0(D, \mathscr{S}^t\mathscr{F}),$$

where C is a general curve linearly equavalent to $t\alpha L$ and D is a 0-cycle of degree $t^2\alpha L^2$. The first term of the second sequence vanishes as $\mathscr{F}(-tL)$ is negative. Hence $h^0(C, \mathscr{S}^t\mathscr{F})$ is bounded by

$$\begin{aligned} t^2\alpha(\operatorname{rank}\mathscr{S}^t\mathscr{F})L^2 &\equiv \frac{\alpha t^{r+1}}{(r-1)!}L^2 \\ &\equiv \frac{t^{r+1}}{6(r+1)!r}\left\{(r-1)c_1^2(\mathscr{E}) - 2rc_2(\mathscr{E})\right\} \pmod{O(t^r)}. \end{aligned}$$

This shows $H^0(S, \mathscr{S}^t\mathscr{F}(-t\alpha L))$ is non-zero whenever t is very large in view of the first exact sequence, and thus such a non-zero global section gives the inclusion $\mathscr{O}_S(t\alpha L) \hookrightarrow \mathscr{S}^t\mathscr{F}$. Similarly, the second case will yield $H^0(S, \mathscr{S}^t\mathscr{F}^*(-t\alpha L)) \neq 0$.

(2). It suffices to consider the following cases:

(a) If $\mathscr{S}^t\mathscr{F}$ contains $\mathscr{O}_S(t\alpha L)$, then

$$\delta_D(\mathscr{E}_1^D) - \delta_D(\mathscr{E}) \geq \alpha LD.$$

(b) If $\mathscr{S}^t\mathscr{F}^*$ contains $\mathscr{O}_S(t\alpha L)$, then

$$\delta_D(\mathscr{E}_1^D) - \delta_D(\mathscr{E}) \geq \frac{1}{r}\left\{\delta_D((\mathscr{E}^*)_1^D) - \delta_D(\mathscr{E}^*)\right\} \geq \frac{\alpha LD}{r}.$$

This completes the proof. □

Corollary 2.4.3. Let $\mathscr{E}$ be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and $H_1, \ldots, H_{n-2}$ be ample Cartier divisors. Let D be a nef Cartier divisor on X. Assume that $H_1 \ldots H_{n-2}D$ is not numerically trivial. If $\mathscr{E}$ is $(H_1, \ldots, H_{n-2}, D)$-semistable, then

$$(r-1)c_1^2(\mathscr{E})H_1 \ldots H_{n-2} \leq 2rc_2(\mathscr{E})H_1 \ldots H_{n-2}.$$

Proof. Suppose the contrary. We may assume that $\mathscr{E}$ is a vector bundle in codimension 2 by taking double dual. Fix an ample divisor H_0 such that $\mathscr{E}(-\delta + H_0)$ and $\mathscr{E}^*(-\delta(\mathscr{E}) + H_0)$ are both ample. Let H be an arbitrary ample divisor. Then there exist positive integer $m_1, \ldots, m_{n-2}$ depending on H such that $H|_S$-semistable filtration of $\mathscr{E}|_S$ coincides with the restriction of $(H_1, \ldots, H_{n-2}, H)$-semistable filtration of $\mathscr{E}$ to a generic complete intersection surface $S = (m_1H_1)\ldots(m_{n-2}H_{n-2})$.

By Corollary 2.4.3, we have

$$\begin{aligned}\delta(\mathscr{E}_1^{(\mathfrak{B},H)})SH - \delta(\mathscr{E}^{(\mathfrak{B},H)})SH &= \delta_H((\mathscr{E}|_S)_1^H) - \delta_H((\mathscr{E}|_S)^H)\\ &\geq c\left\{(r-1)c_1^2(\mathscr{E}|_S) - 2rc_2(\mathscr{E}|_S)\right\}(H,H_0)_S/(H_0^2)_S\\ &= c\left\{(r-1)c_1^2(\mathscr{E}) - 2rc_2(\mathscr{E})\right\}HH_0S/H_0^2S,\end{aligned}$$

where $\mathfrak{B} = (H_1,\dots,H_{n-2})$ and c is a constant. Therefore, by dividing out both sides by $m_1\dots m_{n-2}$, we obtain the inequality

$$\delta_{(\mathfrak{B},H)}(\mathscr{E}_1^{(\mathfrak{B},H)}) \geq \delta_{(\mathfrak{B},H)}(\mathscr{E}) + cHH_0H_1\dots H_{n-2}.$$

By the continuity of the function $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ on a segment joining $(\mathfrak{B},D)$ and $(\mathfrak{B},H)$, we have

$$\delta_{(\mathfrak{B},D)}(\mathscr{E}_1^{(\mathfrak{B},D)}) \geq \delta_{(\mathfrak{B},H)}(\mathscr{E}) + cDH_0H_1\dots H_{n-2} > \delta_{(\mathfrak{B},D)}(\mathscr{E}),$$

a contradiction. □

Corollary 2.4.4. Let $\mathscr{E}$ be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and $H_1,\dots,H_{n-2}$ be ample Cartier divisors. If

$$\{(r-1)c_1^2(\mathscr{E}) - 2rc_2(\mathscr{E})\}H_1\dots H_{n-2} > 0,$$

then $\mathscr{E}$ is $(H_1,\dots,H_{n-2},D)$-unstable for any non-zero nef divisor D.

2.5. Semistability in positive and mixed characteristic.

2.5.1. *Semistability in positive characteristic.* Let C be a smooth complete curve over an algebraically closed field k of characteristic $p > 0$.

Definition 2.5.1. A vector bundle $\mathscr{E}$ on C is said to be **strongly semistable** if, for every positive integer s, $(F^s)^*\mathscr{E}$ is semistable.

Remark 2.5.1. If C is an elliptic curve, it's known that every semistable bundle is strongly semistable, but that is not the case when $g(C) \geq 2$.

Proposition 2.5.1. If $\mathscr{E}$ is strongly semistable on C, then $f^*\mathscr{E}$ is semistable for any surjective k-morphism $f : C' \to C$.

Proof. Let C'' be a smooth model of the seperable closure of C. The natural projection $C' \to C''$ is pure inseparable and hence $C' = F^{-s}C''$ for some non-negative integer s (Proposition IV 2.5 of [Har77]). Thus we get the commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{F^s} & C'' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ F^{-s}C & \xrightarrow{F^s} & C \end{array}$$

Since $\mathscr{E}$ is strongly semistable, we have $(F^s)^*$ is semistable on $F^{-s}C$, and thus $f^*\mathscr{E} = g^*(F^s)^*\mathscr{E}$ is also semistable by Lemma 2.2.1 as g is seperable. □

Remark 2.5.2. The Theorem 2.2.1 and its corollaries still hold in positive characteristic if the “semistability'' is subsituted by “strong semistability”.

2.5.2. *Semistability in mixed characteristic.* Let X be a smooth projective variety over a noetherian integral domain R of characteristic zero and $\mathscr{E}$ be a torsion-free sheaf on X. Fix $\mathfrak{A} \in \overline{\mathrm{NA}}(X/R)^{n-1}_{\mathbb{Q}}$, where n is the relative dimension of X. Then the set of geometric points $t \in \operatorname{Spec} R$ such that $\mathscr{E}_t/X_t$ is $\mathfrak{A}$-semistable forms an open subset.

On the contrary, we know very little about the strong semistability of the reductions of a semistable sheaves.

Question 2.5.1. Let C be an irreducible smooth projective curve over a noetherian integral domain R of characteristic zero. Assume that a locally free sheaf $\mathscr{E}$ on C is $\mathfrak{A}$-semistable on the generic fibre C_*. Let S be the set of primes of positive characteristic on $\operatorname{Spec} R$ such that $\mathscr{E}$ is strongly semistable. Is S a dense subset of $\operatorname{Spec} R$?

2.6. **Generic semipositive theorem for cotangent bundle.** From now on, all varieties are defined over an algebraically closed field k of characteristic 0. Let X be a normal projective variety of dimension n.

Definition 2.6.1. Let $\mathfrak{B} \in \overline{\mathrm{NA}}(X)^{n-2}_{\mathbb{Q}}$.

(1) A torsion-free sheaf $\mathscr{E}$ on X is said to be **generically $\mathfrak{B}$-seminegative** if, for every numerically effective $\mathbb{Q}$-Cartier divisor D on X, its maximal $(\mathfrak{B}, D)$-destabilizing subsheaf $\mathscr{E}_1$ satisfies $\delta_{(\mathfrak{B},D)}(\mathscr{E}_1) < 0$.
(2) A torsion-free sheaf $\mathscr{E}$ on X is said to be **generically $\mathfrak{B}$-semipositive** if $\mathscr{E}^*$ is generically $\mathfrak{B}$-seminegative.

Lemma 2.6.1. Let $\mathscr{E}$ be a torsion-free sheaf on X and

$$0 = \mathscr{E}_0 \subseteq \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \mathscr{E}$$

be the $(\mathfrak{B}, D)$-semistable filtration of $\mathscr{E}$ and put $\alpha_i = \delta_{(\mathfrak{B},D)}(\mathscr{E}_i/\mathscr{E}_{i-1})$. Then $\alpha_1 > \cdots > \alpha_s \geq 0$ for every $D \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}$ if $\mathscr{E}$ is generically $\mathfrak{B}$-semipositive.

Proof. It follows from the definition. □

Theorem 2.6.1. Let $\mathfrak{B} = (H_1, \ldots, H_{n-2}) \in \mathrm{NA}(X)^{n-2}_{\mathbb{Q}}$ and $\mathscr{E}$ be a generically $\mathfrak{B}$-semipositive torsion-free sheaf on X. Then

$$c_2(\mathscr{E}) H_1 \ldots H_{n-2} \geq 0$$

holds.

Theorem 2.6.2. Let $\mathfrak{B} = (H_1, \ldots, H_{n-2}) \in \mathrm{NA}(X)_{\mathbb{Q}}^{n-2}$. Then the torsion-free sheaf $\rho_* \Omega^1_{X'}$ is generically $\mathfrak{B}$-semipositive unless X is uniruled, where $\rho\colon X' \to X$ denotes an arbitrary resolution.

3. Results

3.1. Semipositivity of $3c_2 - c_1^2$.

Proposition 3.1.1. Let X be a non-uniruled, normal projective variety of dimension n with $\mathbb{Q}$-Cartier canonical divisor K_X which is nef. Let $\mathfrak{B} \in \mathrm{NA}(X)_{\mathbb{Q}}^{n-2}$ such that $K_X^2|\mathfrak{B}|$ is positive. Then

$$\{3c_2(\mathscr{E}) - c_1(\mathscr{E})^2\}|\mathfrak{B}| \geq 0,$$

where $\mathscr{E} = \rho_* \Omega^1_{X'}$ and $\rho\colon X' \to X$ is an arbitrary resolution.

3.2. Non-negativity of the Kodaira dimension of minimal threefolds.

3.2.1. *The Gorenstein case.*

Theorem 3.2.1. Let X be a normal projective Gorenstein threefold with only canonical singularities (X is Gorenstein if and only if K_X is a Cartier divisor). Assume K_X is nef. Then the Euler characteristic $\chi(X, \mathscr{O}_X)$ is non-negative. In particular, either $h^0(X, \mathscr{O}_X(K_X))$ or $h^1(X, \mathscr{O}_X)$ is non-zero, and thus $\kappa(X) \geq 0$.

3.2.2. *The K_X^2 is numerically non-trivial case.*

Theorem 3.2.2. Let X be a normal projective Gorenstein threefold with only isolated singularities. Assume the $\mathbb{Q}$-Cartier divisor K_X is nef and K_X^2 is numerically non-trivial. Then $\kappa(X) \geq 0$.

REFERENCES

[BS58] Armand Borel and Jean-Pierre Serre. Le théorème de Riemann-Roch. *Bull. Soc. Math. France*, 86:97–136, 1958.

[GD71] A. Grothendieck and J. A. Dieudonné. *Éléments de géométrie algébrique. I*, volume 166 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1971.

[Har77] Robin Hartshorne. *Algebraic geometry*, volume No. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977.

[HN75] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.*, 212:215–248, 1974/75.

[Ish14] Shihoko Ishii. *Introduction to singularities*. Springer, Tokyo, 2014.

[Kob87] Shoshichi Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ; Princeton University Press, Princeton, NJ, 1987. Kanô Memorial Lectures, 5.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Miy87] Yoichi Miyaoka. The Chern classes and Kodaira dimension of a minimal variety. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 449–476. North-Holland, Amsterdam, 1987.

[MR82] V. B. Mehta and A. Ramanathan. Semistable sheaves on projective varieties and their restriction to curves. *Math. Ann.*, 258(3):213–224, 1981/82.