Bridgeland stability and dHYM metric of line bundles on surfaces

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It's a fundamental principle which has guided much of the research in complex geometry since the late 20-th century: stable objects in the algebraic geometry should correspond to extremal objects in the differential geometry.

This philosophy was motivated by the relations between the slope stability and the existence of Hermitian-Einstein metric.

The curve case was established by Narasimhan-Seshadri [NS65].

Theorem 1.1

A holomorphic vector bundle $\mathcal{E}$ on a compact Riemann surface is stable if and only if there is an irreducible Hermitian-Einstein metric on $\mathcal{E}$.

The further work on this topic is summarized as follows:

- Donaldson gave a new proof of Narasimhan-Seshadri's result in [Don83], and then he proved the surface case in [Don85].
- The compact Kähler manifold case was proved by Uhlenbeck-Yau in [UY86].
- The non-Kähler case was proved by Li-Yau in [LY87].
- The Higgs bundle case was proved by Simpson in [Sim88].
- Yau conjectured that the existence of Kähler-Einstein metric on Fano manifold should be equivalent to some algebro-geometric stability conditions, which was solved in [CDS15a], [CDS15b] and [CDS15c].

On the other hand, going back to the work of Douglas, and Thomas-Yau ([TY02]), it has long been conjectured that the existence of special Lagrangians or solutions of dHYM equation is equivalent to a purely algebraic notion of stability.
The present version of this folklore conjecture is

Conjecture 1.1

A line bundle $\mathcal{L}$ admits a dHYM metric if and only if it is stable in the sense of Bridgeland as an object in $D^b(X)$.

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- In the first part we will introduce results in [AM16], in which the authors proposed a conjecture about Bridgeland stability of line bundle on a smooth projective complex surface S, and proved some special cases of this conjecture. In general case, this conjecture is still open.
- In the second part we will give a brief review of backgrounds about deformed Hermitian-Yang-Mills equation and twisted ampleness criterion.
- Finally we give a counter-example which shows that Bridgeland stability of line bundle and the existence of dHYM metric does not coincide. This counter-example is due to [CS22] originally, and we point out it's a general phenomenon.

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Let S be a smooth projective complex surface. Let ω be an ample $\mathbb{R}$-divisor on S and B be a $\mathbb{R}$-divisor on S. In this talk we focus on the Bridgeland stability $(\mathscr{B}_{\omega,B}, Z_{\omega,B})$, where

$$
\begin{aligned}
Z_{\omega,B}(\mathcal{E}) &= -\int_X e^{-\sqrt{-1}\omega}\operatorname{ch}^B(\mathcal{E}) \\
&= -\operatorname{ch}_2(\mathcal{E}) + \operatorname{ch}_1(\mathcal{E})\cdot B - \frac{\operatorname{ch}_0(\mathcal{E})}{2}(B^2-\omega^2)+ \\
&\quad \sqrt{-1}\left(\operatorname{ch}_1(\mathcal{E})\cdot\omega - \operatorname{ch}_0(\mathcal{E})\omega\cdot B\right)
\end{aligned}
$$

and $\mathscr{B}_{\omega,B}$ is given by the torsion pair

$$
\begin{aligned}
\mathscr{T}_{\omega,B} &= \{\mathcal{E}\in \operatorname{Coh}(S) \mid \mu_{\omega,B,\min}(\mathcal{E}) > 0\} \\
\mathscr{F}_{\omega,B} &= \{\mathcal{E}\in \operatorname{Coh}(S) \mid \mu_{\omega,B,\max}(\mathcal{E}) \le 0\}.
\end{aligned}
$$

- For any line bundle $\mathcal{L}$, it's always stable in the sense of slope stability.
- Since every line bundle can be regarded as a complex of sheaves which centered at zero degree, a natural question is, if it lies in the heart of a Bridgeland stability condition $\omega_{\omega,B}$, does it always stable with respect to $\sigma_{\omega,B}$?

In [AM16], the authors proposes the following conjecture.

Conjecture 1.1

Let $\sigma_{\omega,B}$ be a Bridgeland stability such that the line bundle $\mathcal{L}$ lies in the heart of $\sigma_{\omega,B}$. Then the only objects that could destabilize $\mathcal{L}$ are line bundles of the form $\mathcal{L}(-C)$, where C is a curve of negative self-intersection.

The main result of [AM16] shows that Conjecture 1.1 holds true in the following cases:

- If S does not have any curves of negative self-intersection. (Theorem 3.1)
- If the Picard rank of S is two and there exists only one irreducible curve of negative self-intersection. (Theorem 4.1)

Definition 2.1

Given two objects $\mathcal{E}, \mathcal{B} \in D^b(S)$, with $\mathcal{B}$ is Bridgeland stable for at least one stability condition.

- The *numerical wall* $\mathscr{W}(\mathcal{E}, \mathcal{B})$ is defined as

$$\{\sigma = (\mathscr{B}, Z) \mid (\mathrm{Re}Z(\mathcal{E}))(\mathrm{Im}Z(\mathcal{B})) - (\mathrm{Re}Z(\mathcal{B}))(\mathrm{Im}Z(\mathcal{E})) = 0\}.$$

- If at some $\sigma \in \mathscr{W}(\mathcal{E}, \mathcal{B})$ we have $\mathcal{E} \subseteq \mathcal{B}$ in $\mathscr{B}$, we say that $\mathscr{W}(\mathcal{E}, \mathcal{B})$ is a *weakly destabilizing wall* for $\mathcal{B}$.
- If at some $\sigma \in \mathscr{W}(\mathcal{E}, \mathcal{B})$ we have $\mathcal{E} \subseteq \mathcal{B}$ in $\mathscr{B}$ and $\mathcal{B}$ is Bridgeland σ-semistable, we say that $\mathscr{W}(\mathcal{E}, \mathcal{B})$ is an *actually destabilizing wall* for $\mathcal{B}$.

In order to study the Bridgeland stability of the line bundles on S, the following lemma shows it suffices to study the Bridgeland stability of $\mathcal{O}_S$, by the action of tensoring stability condition.

Lemma 2.1 ([AM16, Lemma 3.1])

Let $\mathcal{L} = \mathcal{O}_S(D_1)$ be a line bundle and $\sigma_{\omega,D}$ be a Bridgeland stability condition. Then $\mathcal{E}$ destabilizes $\mathcal{L}$ at $\sigma_{\omega,D}$ if and only if $\mathcal{O}_S(-D_1) \otimes \mathcal{E}$ destabilizes $\mathcal{O}_S$ at $\sigma_{\omega,D-D_1}$

In order to study the Bridgeland stability of structure sheaf $\mathcal{O}_S$, firstly let's study the possible subojects of $\mathcal{O}_S$, which is completely different from the case of slope stability.

Lemma 2.2 ([AM16, Lemma 4.1])

Let σ be a Bridgeland stability condition and $0 \to \mathcal{E} \to \mathcal{O}_S \to \mathcal{Q} \to 0$ be a short exact sequence in heart of σ. Then $\mathcal{E}$ is a torsion-free sheaf and $\mathcal{H}^0(\mathcal{Q})$ is a quotient of $\mathcal{O}_S$ of rank zero. In particular, the kernel of the map $\mathcal{O}_S \to \mathcal{H}^0(\mathcal{Q})$ is an idea sheaf $\mathcal{I}_Z(-C)$ for some effective curve C and some zero-dimensional subvariety Z (with C or Z possibly zero).

Proof.

Consider the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{H}^{-1}(\mathcal{E}) \to 0 \to \mathcal{H}^{-1}(\mathcal{Q}) \to \mathcal{E} \to \mathcal{O}_S \to \mathcal{H}^0(\mathcal{Q}) \to 0.$$

This shows $\mathcal{H}^{-1}(\mathcal{E}) = 0$ and thus $\mathcal{E}$ must be a sheaf. Since $\mathcal{O}_S \to \mathcal{H}^0(\mathcal{Q}) \to 0$ is exact, we must have the kernel of this map is trivial or an idea sheaf of the form $\mathcal{I}_Z(-C)$, where C is an effective curve and Z is a zero-dimensional subvariety.
If the kernel is trivial, that is, $\mathcal{H}^0(\mathcal{Q}) \cong \mathcal{O}_S$, then we would have $\mathcal{H}^{-1}(\mathcal{Q}) \cong \mathcal{E}$. However, $\mathcal{H}^{-1}(\mathcal{Q}) \in \mathscr{F}, \mathcal{E} \in \mathscr{T}$ and $\mathscr{F} \cap \mathscr{T} = \{0\}$. Therefore, in this case, both $\mathcal{H}^{-1}(Q)$ and $\mathcal{E}$ would have to be zero. This shows the kernel of $\ker\{\mathcal{O}_S \to \mathcal{H}^0(\mathcal{Q})\} = \mathcal{I}_Z(-C)$. Since $\mathcal{E}$ is an extension of torsion-free sheaves $\mathcal{I}_Z(-C)$ and $\mathcal{H}^{-1}(\mathcal{Q})$, it's also a torsion-free sheaf. □

- Now let's compute the numerical walls of $\mathcal{O}_S$.
- One of the features that makes stability conditions well suited to compututations is its decomposition into well behaved 3-slices. For convenience we rescale ω such that $\omega^2 = 1$ and choose an $\mathbb{R}$-divisor G with $G \cdot \omega = 0$. We define the 3-slice as follows

$$S_{\omega,G} = \{\sigma_{t\omega,s\omega+uG} \mid t, s, u \in \mathbb{R}, t > 0\}.$$

- For $\sigma_{t\omega,s\omega+uG} \in S_{\omega,G}$, we denote the heart of $\sigma_{t\omega,s\omega+uG}$ by $\mathscr{B}_{t,s,u}$, and denote the slope function of $\sigma_{t\omega,s\omega+uG}$ by $\beta_{t,s,u}$ for convenience.

Lemma 2.3 ([AM16, Remark 4.6])

The short exact sequence $0 \to \mathcal{E} \to \mathcal{O}_S \to \mathcal{Q} \to 0$ in $\mathscr{B}_{t,s,u}$ if and only if

$$\mu_\omega(\overline{J}) < s < \mu_\omega(\underline{K}) < 0,$$

where

- $$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E}$$

  is the Harder-Narasimhan filtration of $\mathcal{E}$ for slope stability and $\underline{K} = \mathcal{E}/\mathcal{E}_{n-1}$.

- $$0 = \mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_m = \mathcal{H}^{-1}(\mathcal{Q})$$

  is the Harder-Narasimhan filtration of $\mathcal{H}^{-1}(\mathcal{Q})$ for slope stability and $\overline{J} = \mathscr{F}_1$.

Proof.

First of all, if $\mathcal{O}_S$ lies in $\mathscr{B}_{t,s,u}$, we have

$$\mu_{t\omega,s\omega+uG}(\mathcal{O}_S) > 0,$$

which is equivalent to $s < 0$.
For $\mathcal{E}$ to be an object of $\mathscr{B}_{t,s,u}$, we must have $s < \mu_\omega(\underline{K})$ and for $\mathcal{H}^{-1}(\mathcal{Q})$ to be an objects of $\mathscr{B}_{t,s,u}$, we must have $s \geq \mu_\omega(\overline{J})$.
The last condition $\mu_\omega(\underline{K}) < 0$ originates from the fact

$$\mu_\omega(\underline{K}) \leq \mu_\omega(\mathcal{E}) < 0,$$

where $\mu_\omega(\mathcal{E}) < 0$ since $\mathcal{E}$ is an extension of $\mathcal{I}_Z(-C)$ by $\mathcal{H}^{-1}(\mathcal{Q})$ for some effective curve C and zero-dimensional subvariety Z and $\mu_\omega(\mathcal{H}^{-1}(\mathcal{Q})) \leq \mu_\omega(\overline{J}) \leq s < 0$. □

- Let $\mathcal{E}$ be a torsion-free sheaf with $\mathrm{ch}(\mathcal{E}) = (r, \mathrm{ch}_1(\mathcal{E}), c)$. We may write
$$\mathrm{ch}_1(\mathcal{E}) = d_h\omega + d_g G + \alpha,$$
where $\alpha \cdot \omega = \alpha \cdot G = 0$.
- For Bridgeland stability $\sigma_{t\omega, s\omega+uG}$, we have
$$\begin{aligned} Z_{t,s,u}(\mathcal{E}) =& (-c + sd_h - ud_g - \frac{r^2}{2}(s^2 - u^2 - t^2)) \\ &+ \sqrt{-1}(td_h - rst). \end{aligned}$$
- The equation of the numerical wall $\mathscr{W}(\mathcal{E}, \mathcal{O}_S)$ is
$$-d_h(s^2 + t^2 + u^2) + 2d_g su + 2cs = 0.$$

- At each fixed u, Maciocia showed in [Mac14] that all walls for $\mathcal{O}_S$ in the plane $\prod_u$ are nested semicircles centered on the s-aixs.
- Thus, given two subojects $\mathcal{E}_1$ and $\mathcal{E}_2$ and a fixed value u, we have $\mathscr{W}(\mathcal{E}_1, \mathcal{O}_S)$ and $\mathscr{W}(\mathcal{E}_2, \mathcal{O}_S)$ are both semicircles, with one of them inside the other one, unless they are equal.
- Now we may think $S_{\omega,G}$ spaces as being extended to the $t = 0$ plane, and we study these quadrics by studying their intersection with $t = 0$:

$$-d_h(s^2 + u^2) + 2d_g su + 2cs = 0. \tag{2.1}$$

Since the walls are semicircles in $\prod_u$ for any u, knowing where the wall is at $t = 0$ would tell us where the wall is at any $t > 0$. The discriminant of (2.1) is

$$\Delta = 4(d_g^2 - d_h^2)$$

and (2.1) can be written as

$$-d_h(s + \frac{d_g}{d_h}u)^2 + \frac{\Delta}{4d_h}u^2 + 2cs = 0.$$

Note that $\mathcal{E} \subseteq \mathcal{O}_S$ in $\mathscr{B}_{t,s,u}$ implies $s < 0$ and by proof in Lemma 2.3 we have $d_h = \mu_\omega(\mathcal{E}) < 0$. Thus the all possible case of weakly destabilizing walls of $\mathcal{O}_S$ are listed as follows:

- For $\Delta = 0$, the parabola case, it can only be a weakly destabilizing wall if $c \geq 0$.
- For $\Delta < 0$, the ellipse case, it can only be a weakly destabilizing wall if $c > 0$.
- For $\Delta > 0$, the hyperbola case, there are three cases given by $c = 0, c > 0$ and $c < 0$.

In this section we prove part of the following theorem to show the basic ideas of the more general case.

Theorem 3.1 ([AM16, Proposition 5.4])

Let S be a smooth projective complex surface and ω, B as before. If S does not contain any curves of negative self-intersection and $\sigma_{\omega,B}$ is a Bridgeland stability condition such that $\mathcal{O}_S \in \mathscr{B}_{\omega,B}$, then $\mathcal{O}_S$ is stable with respect to $\sigma_{\omega,B}$.

- Let $\sigma_{t\omega, s\omega+uG}$ be a Bridgeland stability in the 3-slice $S_{\omega,G}$.
- If a suboject $\mathcal{E} \subseteq \mathcal{O}_S$ in $\mathscr{B}_{t,s,u}$ is of rank one, then by Lemma 2.2 we have $\mathcal{E}$ must be equal to $\mathcal{I}_Z(-C)$ for some effective curve C and some zero-dimensional scheme Z with C or Z possibly 0.
- In this section we prove that if $C = 0$ or $C^2 \geq 0$, then $\mathcal{O}_S(-C)$ does not weakly destabilize $\mathcal{O}_S$. This is a special case of Theorem 3.1.

Proposition 3.1

$\mathcal{I}_Z$ does not weakly destabilize $\mathcal{O}_S$.

Proof.

Let $i \colon Z \to X$ be a zero-dimensional scheme of length $\ell(Z)$. Then the Chern character of $\mathcal{I}_Z$ is

$$\mathrm{ch}(\mathcal{I}_Z) = (1, 0, -\ell(Z)),$$

and thus we have

$$\beta_{t,s,u}(\mathcal{I}_Z) = \frac{-2\ell(Z) + s^2 - u^2 - t^2}{-2st}.$$

Continuation.

On the other hand, we have

$$\beta_{t,s,u}(\mathcal{O}_S) = \frac{s^2 - u^2 - t^2}{-2st}.$$

Therefore, when $\mathcal{O}_S \in \mathscr{B}_{t,s,u}$, we have $s < 0$ and

$$\beta_{t,s,u}(\mathcal{I}_Z) < \beta_{t,s,u}(\mathcal{O}_S).$$

This means that $\mathcal{I}_Z$ does not weakly destabilize $\mathcal{O}_S$ whenever $\mathcal{O}_S \in \mathscr{B}_{t,s,u}$. □

Proposition 3.2 ([AM16, Proposition 5.1])

If $C^2 \geq 0$, then $\mathcal{I}_Z(-C)$ does not weakly destabilize $\mathcal{O}_S$.

Proof.

Suppose $C = c_h\omega + c_G G + \alpha_C$ with $\alpha_C \cdot \omega = \alpha_C \cdot G = 0$. Then $C^2 = c_h^2 - c_g^2 + \alpha_C^2$, and $\alpha_C^2 \leq 0$ by the Hodge index theorem. Theorefore, if $C^2 \geq 0$, then $c_h^2 - c_g^2 \geq 0$. Note that Chern character of $\mathcal{I}_Z(-C)$ is

$$\mathrm{ch}(\mathcal{I}_Z(-C)) = (1, -C, \frac{1}{2}C^2 - \ell(Z)).$$

Then the equation for the wall $\mathscr{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$ simplifies to

$$c_h(s^2 + t^2 + u^2) - 2c_g su + (c_h^2 - c_g^2)s + \alpha_C^2 s - 2\ell(Z)s = 0.$$

Continuation.

- If $c_h^2 - c_g^2 > 0$, then the wall at $t = 0$ is an ellipse going through $(0, 0)$ and

$$P_W = \frac{C^2 - 2\ell(Z)}{c_g^2 - c_h^2}(c_h, c_g),$$

where these are the two points where the tangent line is vertical. Therefore, the s-value of any point on the ellipse is between 0 and the s-value of P_W, which is

$$\frac{C^2 - 2\ell(Z)}{c_g^2 - c_h^2} c_h = -c_h + \frac{\alpha_C^2 - 2\ell(Z)}{c_g^2 - c_h^2} c_h \geq -c_h.$$

But $\mathcal{I}_Z(-C) \in \mathscr{B}_{t,s,u}$, we have that $s < -c_h$ and therefore $\mathcal{I}_Z(-C)$ cannot weakly destabilize $\mathcal{O}_S$.

Continuation.

This can be seen as follows (Figure 9 in [AM16])

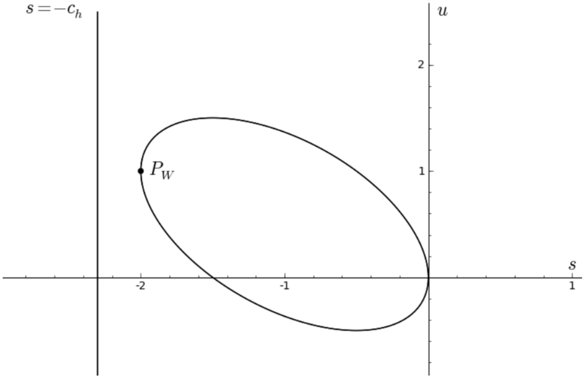

Figure 9: $\mathcal{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$ at $t=0$ when $c_h^2 - c_g^2 > 0$

Continuation.

- If $c_h^2 - c_g^2 = 0$, then $C^2 \geq 0$ implies $C^2 = 0$ and $\mathrm{ch}_2(\mathcal{I}_Z(-C))) = -\ell(Z) < 0$. As we list all possible weakly destabilizing wall for $\mathcal{O}_S$, this wall cannot be a weakly destabilizing wall.

□

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Theorem 4.1 ([AM16, Proposition 5.12])

Let S be a smooth projective complex surface of Picard rank two. Assume that the effective cone of S is generated by C_1 and C_2 such that $C_1 \cdot G > 0$ and C_1 is the only irreducible curve in S of negative self-intersection. Then $\mathcal{O}_S$ is only destabilized by $\mathcal{O}_S(-C_1)$.

Let X be a smooth projective complex variety and ω be an ample $\mathbb{R}$-divisor on X.

Definition 1.1

Let α be a real $(1,1)$-form on X. The *deformed Hermitian-Yang-Mills (dHYM) equation* seeks a function $\phi\colon X \to \mathbb{R}$ such that $\alpha_\phi = \alpha + \sqrt{-1}\partial\overline{\partial}\phi$, which satisfies

$$\mathrm{Im}(e^{-\sqrt{-1}\widehat{\theta}}(\omega + \sqrt{-1}\alpha_\phi)^n) = 0,$$

where

$$\int_X (\omega + \sqrt{-1}\alpha_\phi)^n \in \mathbb{R}_{>0}\, e^{\sqrt{-1}\widehat{\theta}}.$$

Remark 1.1

If we fix a point $p \in X$ and choose a holomorphic coordinate $\{z^i\}$ centered at p such that

$$\omega = \sqrt{-1}\sum_i \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^i, \quad \alpha_\phi = \sqrt{-1}\sum_i \lambda_i \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^i,$$

then the dHYM equation can be written as

$$\Theta_\omega(\alpha_\phi) = \widehat{\theta} \pmod{2\pi},$$

where $\Theta_\omega(\alpha_\phi) = \sum_i \arctan(\lambda_i)$ is called the *Lagrangian phase operator*.

Definition 1.2

Let $\mathcal{L} \to (X, \omega)$ be a line bundle. A Hermitian metric h on $\mathcal{L}$ is called a dHYM metric with respect to ω if the Chern curvature Θ_h satisfies

$$\operatorname{Im}\left(e^{-\sqrt{-1}\widehat{\theta}}\left(\omega - \frac{\Theta_h}{2\pi}\right)^n\right) = 0,$$

where

$$\int_X \left(\omega - \frac{\Theta_h}{2\pi}\right)^n \in \mathbb{R}_{>0}\, e^{\sqrt{-1}\widehat{\theta}}.$$

- The dHYM metric on a line bundl $\mathcal{L}$ is a special case of the dHYM equation. If we choose real $(1,1)$-form $\alpha = \mathrm{ch}_1(\mathcal{L})$, then a solution of dHYM equation gives a dHYM metric on $\mathcal{L}$.
- Given a $\mathbb{R}$-divisor B on X, which is called a *B-field* in literature, a dHYM metric with respect to ω and B is a solution of dHYM equation defined by real $(1,1)$-class $\mathrm{ch}_1^B(\mathcal{L})$, where $\mathrm{ch}_1^B(\mathcal{L}) = e^{-B}\,\mathrm{ch}_1(\mathcal{L})$ is the *twisted Chern character*.

- The higher rank version of dHYM equation was proposed by Collins-Yau in [CY18, &8.1]. For a holomorphic vector bundle $\mathcal{E} \to (X, \omega)$, a Hermitian metric h is called a dHYM metric if the Chern curvature Θ_h satisfies

$$\mathrm{Im}\left(e^{-\sqrt{-1}\widehat{\theta}}\left(\omega \otimes \mathrm{id}_{\mathcal{E}} - \frac{\Theta_h}{2\pi}\right)^n\right) = 0,$$

where

$$\int_X \mathrm{tr}_h\left(\omega \otimes \mathrm{id}_{\mathcal{E}} - \frac{\Theta_h}{2\pi}\right)^n \in \mathbb{R}_{>0}\, e^{\sqrt{-1}\widehat{\theta}}$$

and the imaginary part is defined using the metric h.

- There are many fundamental results about dHYM metric for the line bundle, such as [JY17, CY18, CJY20], but the existence of the solution to the higher rank version is still in mystery.

In [CJY20], the authors proved a Nakai-Moishezon type criterion for the existence of dHYM metric of line bundles on Kähler surface, which is also called twisted ampleness criterion in [CLSY23].

Theorem 2.1

Let (X, ω) be a Kähler surface and $\mathcal{L}$ be a line bundle on X such that $\omega \cdot \mathrm{ch}_1(\mathcal{L}) > 0$. Then $\mathcal{L}$ admits a dHYM metric (with respect to ω) if and only if for every curve $C \subseteq X$ we have

$$\operatorname{Im}\left(\frac{Z_C(\mathcal{L})}{Z_X(\mathcal{L})}\right) > 0,$$

where

$$Z_C(\mathcal{L}) = -\int_C e^{-\sqrt{-1}\omega}\,\mathrm{ch}(\mathcal{L}),$$
$$Z_X(\mathcal{L}) = -\int_X e^{-\sqrt{-1}\omega}\,\mathrm{ch}(\mathcal{L}).$$

- In [FYZ23], the authors showed the solution to dHYM equation on a compact Kähler surface (X, ω) always exists on the complement of a finite number of curves of negative self-intersection.
- In particular, the twisted ampleness criterion is satisfied automatically if there is no negative self-intersection curve on X.

It motivates us to consider Question 2.1. In some special cases, this can be checked directly, such as Hirzebruch surface, which also serves as an important example later.

Question 2.1

Whether it suffices to test negative self-intersection curves in twisted ampleness criterion or not.

Proposition 2.1

Let $X = \mathscr{H}_r$ be the Hirzebruch surface and $\{D_1, D_2, D_3, D_4\}$ be the generators of torus-invariant divisors on X. Then

- ([CLS11, Example 4.1.8]) The Picard group is generated by $\{D_1, D_2, D_3, D_4\}$ with relations

$$0 \sim \mathrm{div}(\chi^{e_1}) = -D_1 + D_3$$
$$0 \sim \mathrm{div}(\chi^{e_2}) = rD_1 + D_2 - D_4.$$

- ([CLS11, Proposition 4.3.3]) The effective cone of X is given by

$$\mathrm{Eff}(X) = \{aD_1 + bD_2 \mid a, b \geq 0\}.$$

Proposition 2.2

Let $X = \mathscr{H}_r$ be the Hirzebruch surface and $\{D_1, D_2, D_3, D_4\}$ be the generators of torus-invariant divisors on X. Then

- ([CLS11, Example 6.1.17]) The ample cone of X is given by

$$\operatorname{Amp}(X) = \{\alpha D_1 + \beta D_4 \mid \alpha, \beta > 0\}.$$

- ([CLS11, Example 6.3.6]) The intersection matrix of D_1 and D_2 is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix}.$$

Proposition 2.3

Let $X = \mathscr{H}_r$ be Hirzebruch surface. Then for the twisted ampleness criterion on X, it suffices to test the only negative self-intersection curve.

Proof.

Let $\omega = \alpha D_1 + \beta D_4$ be an ample $\mathbb{R}$-divisor on X and $\mathcal{L} = kD_3 + \ell D_4$ be a line bundle such that $\omega \cdot \mathrm{ch}_1(\mathcal{L}) > 0$. For curve $C \subseteq X$, the twisted ampleness criterion for C can be rewritten as

$$(C \cdot \mathrm{ch}_1(\mathcal{L}))\,(\omega \cdot \mathrm{ch}_1(\mathcal{L})) > \left(\mathrm{ch}_2(\mathcal{L}) - \frac{1}{2}\omega^2\right)(C \cdot \omega)\,. \qquad (3.1)$$

Continuation.

Take $C = D_1$, equation (3.1) gives

$$\ell(\alpha\ell + \beta k + r\beta\ell) > \frac{1}{2}(2k\ell + r\ell^2 - 2\alpha\beta - r\beta^2)\beta,$$

which is equivalent to

$$\alpha\ell^2 + \alpha\beta^2 + \frac{1}{2}(r\beta\ell^2 + r\beta^3) > 0. \tag{3.2}$$

It's clear that equation (3.2) holds for arbitrary $k, \ell \in \mathbb{Z}$ since $\alpha, \beta > 0$. This completes the proof since the twisted ampleness criterion is linear with respect to the intersection with C. □

This counter-example is due to [CS22] originally.

Proposition 4.1

Let $X = \mathscr{H}_1 = \mathrm{Bl}_p \mathbb{P}^2$ and ω be an ample $\mathbb{R}$-divisor on X. Let $\mathcal{L}$ be a line bundle on X with $\omega \cdot \mathrm{ch}_1(\mathcal{L}) > 0$. If $\mathcal{L}$ admits a dHYM metric with respect to ω, then $\mathcal{L}$ is Bridgeland stable at $\sigma_{\omega,0}$. The converse statement is not true.

Proof.

A line bundle $\mathcal{L} \in \mathscr{B}_{\omega,0}$ if and only if $\omega \cdot \mathrm{ch}_1(\mathcal{L}) > 0$, and by Theorem 4.1 it's Bridgeland stable at $\sigma_{\omega,0}$ if and only if

$$\beta(\mathcal{L}(-C)) < \beta(\mathcal{L}), \tag{4.1}$$

where ρ is the slope function given by $\sigma_{\omega,0}$. A direct computation shows that (4.1) is equivalent to

$$\left(C \cdot \mathrm{ch}_1(\mathcal{L}) - \frac{1}{2}C^2\right)(\omega \cdot \mathrm{ch}_1(\mathcal{L})) > \left(\mathrm{ch}_2(\mathcal{L}) - \frac{1}{2}\omega^2\right)(C \cdot \omega), \tag{4.2}$$

where $C \subseteq X$ is the only curve which has negative self-intersection.

Continuation.

On the other hand, the twisted ampleness criterion (Theorem 2.1) shows that $\mathcal{L}$ admits a solution to dHYM equation if and only if

$$(C \cdot \mathrm{ch}_1(\mathcal{L}))\,(\omega \cdot \mathrm{ch}_1(\mathcal{L})) > \left(\mathrm{ch}_2(\mathcal{L}) - \frac{1}{2}\omega^2\right)(C \cdot \omega)\,, \tag{4.3}$$

for all curves $C \subseteq X$. As a consequence, (4.3) implies (4.2), that is, every line bundle admitting a dHYM metric is Bridgeland stable with respect to $\sigma_{\omega,0}$.

Conversely, if we denote the torus-invariant divisors on $\mathrm{Bl}_p\,\mathbb{P}^2 = \mathscr{H}_r$ by $\{D_1, \ldots, D_4\}$, then $C = D_2$ is the only curve which has negative self-intersection.

Continuation.

Let $\omega = \frac{1}{\sqrt{3}}(D_1 + D_4)$ and $\mathcal{L} = \mathcal{O}_X(2D_4)$. Then

$$\begin{aligned} C \cdot \mathrm{ch}_1(\mathcal{L}) &= 0, \\ \frac{1}{2}C^2 &= -\frac{1}{2}, \\ \omega \cdot \mathrm{ch}_1(\mathcal{L}) &= \frac{4}{\sqrt{3}}, \\ \mathrm{ch}_2(\mathcal{L}) &= 2, \\ \frac{1}{2}\omega^2 &= \frac{1}{3}, \\ C \cdot \omega &= \frac{1}{\sqrt{3}}. \end{aligned}$$

This shows $\mathcal{L}$ is Bridgeland stable with respect to $\sigma_{\omega,0}$, but does not admit a dHYM metric with respect to ω.

- A natural question is this counter-example an coincidence or a common phenomenon?
- In [LW24], we prove that it's not an coincidence at not.
- To be precise, for any ample $\mathbb{R}$-divisor ω on Hirzebruch surface $\mathscr{H}_r$, there always exists a line bundle which is Bridgeland stable with respect to $\sigma_{\omega,0}$, but does not admit a dHYM metric with respect to ω.

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