BRIDGELAND STABILITY AND EXISTENCE OF DEFORMED HERMITIAN-YANG-MILLS METRIC ON SURFACE

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Contents

1. Algebraic geometry aspects

1.1. **Bridgeland stability.** Let X be a smooth projective complex variety of dimension n and ω be an ample divisor on X.

Definition 1.1.1 ([Bri07, Proposition 5.3])**.** A *Bridgeland stability* on the derived category $D^b\operatorname{Coh}(X)$ consists of a pair $(\mathscr{A}, Z)$, where $\mathscr{A}$ is a heart of some t-structure, and $Z\colon K(\mathscr{A})\to\mathbb{C}$ is an additive group homomorphism called *central charge* such that

(1) for any $0\neq\mathcal{E}\in\mathscr{A}$, the complex number $Z(\mathcal{E})$ lies in the strict upper half-plane
$$\{re^{\sqrt{-1}\pi\phi}\mid r>0 \text{ and } 0<\phi\leq 1\}\subset\mathbb{C}.$$
(2) Z satisfies the Harder-Narasimhan property with respect to slope function $\mu(\mathcal{E}):=-\frac{\operatorname{Re}Z(\mathcal{E})}{\operatorname{Im}Z(\mathcal{E})}$, that is, for any $0\neq\mathcal{E}\in\mathscr{A}$, there exists a *Harder-Narasimhan filtration*
$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_n=\mathcal{E}$$
of objects $\mathcal{E}_i\in\mathscr{A}$ such that $\mathscr{G}_i=\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable with respect to μ for all $i=1,\dots,n$ with strictly decreasing slopes.
(3) Z satisfies the *support property*, that is,
$$\inf\left\{\frac{|Z(\mathcal{E})|}{\|\operatorname{ch}(\mathcal{E})\|}\mid 0\neq\mathcal{E}\in\mathscr{A}\text{ is semistable}\right\}>0,$$
where $\|\text{-}\|$ is any norm on the numerical Grothendieck group[1].

Definition 1.1.2. Let $\mathscr{A}$ be a heart of some t-structure and $Z\colon K(\mathscr{A})\to\mathbb{C}$ be an additive group homomorphism.

(1) If $(\mathscr{A}, Z)$ only satisfies (1) in Definition 1.1.1, then Z is called a *stability function*.
(2) If $(\mathscr{A}, Z)$ only satisfies (1) and (2) in Definition 1.1.1, then $(\mathscr{A}, Z)$ is called a *stability condition*.

The following two propositions are very useful criterions for the existence of Harder-Narasimhan filtration and support condition.

Proposition 1.1.1. Let $\mathscr{A}$ be a heart of some t-structure and $Z\colon K(\mathscr{A})\to\mathbb{C}$ be a stability function. Assume that

(1) $\mathscr{A}$ is Noetherian;
(2) the image of $\operatorname{Im}Z$ is discrete in $\mathbb{R}$.

Then Harder-Narasimhan filtration exists in $\mathscr{A}$ with respect to Z.

Proposition 1.1.2. A stability condition $(\mathscr{A}, Z)$ satisfies the support condition if and only if there is a quadratic form Q on the numerical Grothendieck group Λ such that

[1]The numerical Grothendieck group is the image of the Chern character map, which is a finitely generated $\mathbb{Z}$-lattice.

(1) the kernel of Z is negative definite with respect to Q;
(2) for any $\mathcal{E} \in \mathscr{P}(\phi)$, we have

$$Q(\mathrm{ch}(\mathcal{E}), \mathrm{ch}(\mathcal{E})) \geq 0.$$

Proof. If $(\mathscr{A}, Z)$ satisfies the support property, then the quadratic form

$$Q(w, w') := \frac{1}{C^2} Z(w)\overline{Z(w')} - \langle w, w' \rangle,$$

satisfies the above properties, where $C = \inf\left\{ \frac{|Z(\mathcal{E})|}{\|\mathrm{ch}(\mathcal{E})\|} \mid 0 \neq \mathcal{E} \in \mathscr{A} \text{ is semistable} \right\}$ and $\langle -, - \rangle$ is an inner product on $\Lambda \otimes \mathbb{R}$. Conversely, if there is a quadratic form Q on Λ with above properties, we may choose $C > 0$ such that

$$\frac{1}{C^2} |Z(w)|^2 - Q(w, w) > 0$$

for all w in the unit ball of $\Lambda \otimes \mathbb{R}$. Then we define

$$\langle w, w' \rangle = \frac{1}{C^2} Z(w)\overline{Z(w')} - Q(w, w').$$

This gives an inner product on $\Lambda \otimes \mathbb{R}$ and for $\mathcal{E} \in \mathscr{P}(\phi)$, we have

$$\begin{aligned} |Z(\mathrm{ch}(\mathcal{E}))|^2 &= C^2 \|\mathrm{ch}(\mathcal{E})\|^2 + C^2 Q(\mathrm{ch}(\mathcal{E}), \mathrm{ch}(\mathcal{E})) \\ &\geq C^2 \|\mathrm{ch}(\mathcal{E})\|^2. \end{aligned}$$

□

Example 1.1.1. *Let C be a smooth projective curve and ω be an ample divisor on C. Then the group homomorphism*

$$Z_\omega = -\int_C e^{-\sqrt{-1}\omega}$$

is a Bridgeland stability on the category of coherent sheaves, which is the heart of the standard t-structure.

Remark 1.1.1. The slope stability on curves is a Bridgeland stability on curves. In higher dimensional case, the existence of Bridgeland stability condition is not that clear. Let X be a smooth projective variety of dimension $n \geq 2$ and ω be an ample divisor on X. The group homomorphism

$$\overline{Z}_\omega(\mathcal{E}) = -\omega^{n-1} \mathrm{ch}_1(\mathcal{E}) + \sqrt{-1}\omega^n \cdot \mathrm{ch}_0(\mathcal{E})$$

is no longer a Bridgeland stability on $\mathrm{Coh}(X)$. Moreover, by [Tod09] there is no Bridgeland stability on $\mathrm{Coh}(X)$ when $n \geq 2$, so it's necessary to consider hearts of other t-structures on $D^b\mathrm{Coh}(X)$ to construct Bridgeland stability on higher dimensional varieties.

1.2. **Bridgeland stability on surface.** In this section, let S be a smooth projective surface over $\mathbb{C}$ and the $\mathscr{B}_\omega = \langle \mathscr{F}_\omega[1], \mathscr{T}_\omega \rangle$ be *tilt heart* given by the following torsion pair

$$\mathscr{T}_\omega = \{\mathcal{E} \in \mathrm{Coh}(S) \mid \mu_{\omega,\min}(\mathcal{E}) > 0\}$$
$$\mathscr{F}_\omega = \{\mathcal{E} \in \mathrm{Coh}(S) \mid \mu_{\omega,\max}(\mathcal{E}) \leq 0\}.$$

The goal of this section is to prove that

$$Z_\omega = -\int_S e^{-\sqrt{-1}\omega}\,\mathrm{ch}$$

is a Bridgeland stability on the tilt heart $\mathscr{B}_\omega$. The following lemma gives a useful description for the tilt heart given by torsion pair in general.

Lemma 1.2.1 ([MS17, Lemma 6.3])**.** Let X be a smooth projective variety and $\mathscr{F}, \mathscr{T}$ be a torsion pair in $\mathrm{Coh}(X)$. Then the tilt heart $\mathscr{B} = \langle \mathscr{F}[1], \mathscr{T} \rangle$ consists of $\mathcal{E} \in D^b\,\mathrm{Coh}(X)$ such that $\mathcal{H}^0(\mathcal{E}) \in \mathscr{T}$, $\mathcal{H}^{-1}(\mathcal{E}) \in \mathscr{F}$ and $\mathcal{H}^i(\mathcal{E}) = 0$ for all $i \neq 0, -1$.

1.2.1. *Stability function.*

Proposition 1.2.1 ([AB13, Corollary 2.1])**.** The group homomorphism

$$\begin{aligned} Z_\omega(\mathcal{E}) &= -\int_S e^{-\sqrt{-1}\omega}\,\mathrm{ch}(\mathcal{E}) \\ &= \left(-\mathrm{ch}_2(\mathcal{E}) + \frac{\omega^2}{2}\mathrm{ch}_0(\mathcal{E})\right) + \sqrt{-1}\omega \cdot \mathrm{ch}_1(\mathcal{E}) \end{aligned}$$

is a stability function on the tilt heart $\mathscr{B}_\omega$.

Proof. By definition, each object $\mathcal{E} \in \mathscr{B}_\omega$ fits into an exact triangle

$$\mathcal{H}^{-1}(\mathcal{E})[1] \to \mathcal{E} \to \mathcal{H}^0(\mathcal{E})$$

with $\mathcal{H}^{-1}(\mathcal{E}) \in \mathscr{F}_\omega$ and $\mathcal{H}^0(\mathcal{E}) \in \mathscr{T}_\omega$. Since Z_ω is additive, it's clear to see $\mathrm{Im}Z_\omega(\mathcal{E}) \geq 0$ by the construction of $\mathscr{F}_\omega$ and $\mathscr{T}_\omega$. Now it suffices to show if $\mathrm{Im}Z_\omega(\mathcal{E}) = 0$, then $\mathrm{Re}Z_\omega(\mathcal{E}) < 0$.

Claim 1.2.1. If $\mathrm{Im}Z_\omega(\mathcal{E}) = 0$, then $\mathcal{H}^{-1}(\mathcal{E})$ is a μ_ω-semistable torsion-free sheaf with $\mu_\omega(\mathcal{H}^{-1}(\mathcal{E})) = 0$ and $\mathcal{H}^0(\mathcal{E})$ is a torsion sheaf with zero-dimensional support.

Proof. Since $\mathcal{H}^{-1}(\mathcal{E}) \in \mathscr{F}_\omega$, we have $\mu_{\omega,\max}(\mathcal{H}^{-1}(\mathcal{E})) \leq 0$. If $\mathcal{H}^{-1}(\mathcal{E})$ is not μ_ω-semistable, by considering its Harder-Narasimhan filtration, we have $\mathrm{Im}Z_\omega(\mathcal{H}^{-1}(\mathcal{E}))$ must be strictly less than zero, a contradiction. This shows $\mathcal{H}^{-1}(\mathcal{E})$ is μ_ω-semistable. Moreover, $\mu_\omega(\mathcal{H}^{-1}(\mathcal{E})) = 0$.

Since $\mathcal{H}^0(\mathcal{E}) \in \mathscr{T}_\omega$, we have $\mu_{\omega,\min}(\mathcal{H}^0(\mathcal{E})) > 0$. If $\mathcal{H}^0(\mathcal{E})$ is not a torsion sheaf with zero-dimensional support, then let's consider the following cases:

(1) If $\mathcal{H}^0(\mathcal{E})$ is a torsion sheaf with one dimensional support, we have $\mathrm{Im}Z_\omega(\mathcal{H}^0(\mathcal{E})) > 0$ since $\mathrm{ch}_1(\mathcal{H}^0(\mathcal{E}))$ is effective, a contradiction.
(2) If $\mathcal{H}^0(\mathcal{E})$ is a torsion-free sheaf, we will also have $\mathrm{Im}Z_\omega(\mathcal{H}^0(\mathcal{E})) > 0$ by the construction of $\mathscr{T}_\omega$, a contradiction.

This completes the proof of the claim. □

Since $\mathcal{H}^0(\mathcal{E})$ is a torsion sheaf with zero-dimensional support, we have

$$\mathrm{Re}Z_\omega(\mathcal{H}^0(\mathcal{E})) = -\operatorname{ch}_2(\mathcal{E}) < 0.$$

On the other hand, since $\omega\cdot\operatorname{ch}_1(\mathcal{H}^{-1}(\mathcal{E})) = 0$, by Hodge index theorem we have $\operatorname{ch}_1^2(\mathcal{H}^{-1}(\mathcal{E})) \le 0$. Then by the Bogomolov-Gieseker inequality, we have $\operatorname{ch}_2(\mathcal{H}^{-1}(\mathcal{E})) \le 0$. Hence,

$$\begin{aligned}\mathrm{Re}Z_\omega(\mathcal{H}^{-1}(\mathcal{E})[1]) &= -\mathrm{Re}Z_\omega(\mathcal{H}^{-1}(\mathcal{E}))\\ &= \underbrace{\operatorname{ch}_2(\mathcal{H}^{-1}(\mathcal{E}))}_{\le 0} - \underbrace{\frac{\omega^2}{2}\operatorname{ch}_0(\mathcal{H}^{-1}(\mathcal{E}))}_{>0} < 0.\end{aligned}$$

This completes the proof. □

1.2.2. *Existence of Harder-Narasimhan filtration.*

Lemma 1.2.2 ([MS17, Lemma 6.17])**.** The tilt heart $\mathscr{B}_\omega$ is a Noetherian category.

Corollary 1.2.1. $(\mathscr{B}_\omega, Z_\omega)$ is a stability condition.

1.2.3. *Support condition.* By Proposition 1.1.2, if we want to show $(\mathscr{B}_\omega, Z_\omega)$ satisfies the support property, it suffices to find a quadratic form Q with certain properties. Actually, it turns out such quadratic form comes from some Bogomolov-Gieseker type inequalities for tilt stability.

Lemma 1.2.3 ([BMT14, Corollary 7.3.3])**.** There exists a constant $C_\omega \ge 0$ such that for every effective divisor D on S, we have

$$C_\omega(\omega D)^2 + D^2 \ge 0.$$

Definition 1.2.1. The *ω-discriminant* is defined as

$$\overline{\Delta}_\omega := (\omega\cdot\operatorname{ch}_1)^2 - 2\omega^2\operatorname{ch}_0\operatorname{ch}_2.$$

Definition 1.2.2. The *(ω, C_ω)-discriminant* is defined as

$$\Delta_\omega^C := \Delta + C_\omega(\omega\cdot\operatorname{ch}_1)^2,$$

where $\Delta = \operatorname{ch}_1^2 - 2\operatorname{ch}_0\operatorname{ch}_2$ is the usual discriminant.

Theorem 1.2.1 ([MS17, Theorem 6.13])**.** If $\mathcal{E} \in \mathscr{B}_\omega$ is ν_ω-semistable, then $\Delta_\omega^C(\mathcal{E}) \ge 0$ and $\overline{\Delta}_\omega(\mathcal{E}) \ge 0$.

Proposition 1.2.2. The (ω, C_ω)-discriminant Δ_ω^C gives the support condition for $(\mathscr{B}_\omega, Z_\omega)$.

Proof. Note that Δ_ω^C is the composition of

$$K(X) \xrightarrow{\operatorname{ch}} H^0(X,\mathbb{R}) \oplus \mathrm{NS}(X)_\mathbb{R} \oplus H^2(X,\mathbb{R}) \xrightarrow{Q_\omega} \mathbb{R},$$

where ch is the Chern character map and the quadratic form Q_ω is given by

$$(r, c, d) \mapsto C_\omega(\omega c)^2 + c^2 - 2rd.$$

By Theorem 1.2.1, for any ν_ω-semistable $\mathcal{E} \in \mathscr{B}_\omega$, one has $Q_\omega(\mathrm{ch}(\mathcal{E})) \geq 0$. Then by Proposition 1.1.2, it suffices to show the kernel of Z_ω is negative definite with respect to Q_ω. If $Z_\omega(\mathcal{E}) = 0$, then $\omega^2 \mathrm{ch}_0(\mathcal{E}) = 2\mathrm{ch}_2(\mathcal{E})$ and $\omega \cdot \mathrm{ch}_1(\mathcal{E}) = 0$. By Hodge index theorem one has $\mathrm{ch}_1^2(\mathcal{E}) \leq 0$, and thus if $\mathrm{ch}(\mathcal{E}) \neq 0$, one has

$$\begin{aligned} Q_\omega(\mathrm{ch}(\mathcal{E})) &= \mathrm{ch}_1^2(\mathcal{E}) - 2\mathrm{ch}_0(\mathcal{E})\mathrm{ch}_2(\mathcal{E}) \\ &= \mathrm{ch}_1^2(\mathcal{E}) - \omega^2 \mathrm{ch}_0^2(\mathcal{E}) < 0. \end{aligned}$$

This completes the proof. □

Corollary 1.2.2. $(\mathscr{B}_\omega, Z_\omega)$ is a Bridgeland stability.

1.3. **Bridgeland stability of line bundles on surface.**

1.3.1. *Introduction.* Let S be a smooth projective complex surface and ω be an ample $\mathbb{R}$-divisor on S. Let B be a $\mathbb{R}$-divisor on S. Then $(\mathscr{B}_{\omega,B}, Z_{\omega,B})$ is a Bridgeland stability on $D^b(S)$, where

$$Z_{\omega,B}(\mathcal{E}) = -\mathrm{ch}_2(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E})\cdot B - \frac{\mathrm{ch}_0(\mathcal{E})}{2}(B^2 - \omega^2) + \sqrt{-1}(\mathrm{ch}_1(\mathcal{E})\cdot\omega - \mathrm{ch}_0(\mathcal{E})\omega\cdot B)$$

and $\mathscr{B}_{\omega,B}$ is given by the torsion pair

$$\begin{aligned} \mathscr{T}_{\omega,B} &= \{\mathcal{E} \in \mathrm{Coh}(S) \mid \mu_{\omega,B,\min}(\mathcal{E}) > 0\} \\ \mathscr{F}_{\omega,B} &= \{\mathcal{E} \in \mathrm{Coh}(S) \mid \mu_{\omega,B,\max}(\mathcal{E}) \leq 0\}. \end{aligned}$$

One of the features that makes stability conditions well suited to compututations is its decomposition into well behaved 3-slices. For convenience we rescale ω such that $\omega^2 = 1$ and choose an $\mathbb{R}$-divisor G with $G\cdot\omega = 0$. Then define the 3-slice

$$S_{\omega,G} = \{\sigma_{t\omega, s\omega+uG} \mid t,s,u \in \mathbb{R}, t > 0\}.$$

It's clear that for any Bridgeland stability σ, one can find a 3-slice $S_{\omega,G}$ such that $\sigma \in S_{\omega,G}$. In [AM16], it studies the Bridgeland stability of the line bundle $\mathcal{L}$ and proposes the following conjecture.

Conjecture 1.3.1. Let $\sigma_{\omega,B}$ be a Bridgeland stability stated as above. Then the only objects that could destabilize a line bundle $\mathcal{L}$ are line bundles of the form $\mathcal{L}(-C)$, where C is a curve of negative self-intersection.

The main result of [AM16] shows that Conjecture 1.3.1 holds true in the following cases:

(1) If S does not have any curves of negative self-intersection. (Theorem 1.3.1)
(2) If the Picard rank of S is two and there exists only one irreducible curve of negative self-intersection. (Theorem 1.3.2)

1.3.2. *Preliminaries on the stabilities of $\mathcal{O}_S$.* In order to study the Bridgeland stability of the line bundles on a smooth projective complex surface S, the following lemma shows it suffices to study the Bridgeland stability of $\mathcal{O}_S$, by the action of tensoring stability condition.

Lemma 1.3.1 ([AM16, Lemma 3.1])**.** Let $\mathcal{L} = \mathcal{O}_S(D_1)$ be a line bundle and $\sigma_{\omega,D}$ be a Bridgeland stability condition. Then $\mathcal{E}$ destabilizes $\mathcal{L}$ at $\sigma_{\omega,D}$ if and only if $\mathcal{O}_S(-D_1)\otimes\mathcal{E}$ destabilizes $\mathcal{O}_S$ at $\sigma_{\omega,D-D_1}$

In order to study the Bridgeland stability of structure sheaf $\mathcal{O}_S$, let's study the subojects of $\mathcal{O}_S$ and their walls.

Definition 1.3.1.

Definition 1.3.2. Given two objects $\mathcal{E},\mathcal{B}\in D^b(S)$, with $\mathcal{B}$ is Bridgeland stable for at least one stability condition.

(1) The *numerical wall* $\mathscr{W}(\mathcal{E},\mathcal{B})$ is defined as
$$\{\sigma = (\mathscr{B}, Z) \mid (\mathrm{Re}Z(\mathcal{E}))(\mathrm{Im}Z(\mathcal{B})) - (\mathrm{Re}Z(\mathcal{B}))(\mathrm{Im}Z(\mathcal{E})) = 0\}.$$
(2) If at some $\sigma\in\mathscr{W}(\mathcal{E},\mathcal{B})$ we have $\mathcal{E}\subseteq\mathcal{B}$ in $\mathscr{B}$, we say that $\mathscr{W}(\mathcal{E},\mathcal{B})$ is a *weakly destabilizing wall* for $\mathcal{B}$.
(3) If at some $\sigma\in\mathscr{W}(\mathcal{E},\mathcal{B})$ we have $\mathcal{E}\subseteq\mathcal{B}$ in $\mathscr{B}$ and $\mathcal{B}$ is Bridgeland σ-semistable, we say that $\mathscr{W}(\mathcal{E},\mathcal{B})$ is an *actually destabilizing wall* for $\mathcal{B}$.

Lemma 1.3.2 ([AM16, Lemma 4.1])**.** Let $\sigma\in\mathrm{Stab}(X)$ be a Bridgeland stability condition and $0\to\mathcal{E}\to\mathcal{O}_S\to\mathcal{Q}\to 0$ be a short exact sequence in heart of σ. Then $\mathcal{E}$ is a torsion-free sheaf and $\mathcal{H}^0(\mathcal{Q})$ is a quotient of $\mathcal{O}_S$ of rank zero. In particular, the kernel of the map $\mathcal{O}_S\to\mathcal{H}^0(\mathcal{Q})$ is an idea sheaf $\mathcal{I}_Z(-C)$ for some effective curve C and some zero-dimensional subvariety Z (with C or Z possibly zero).

Proof. Consider the long exact sequence in cohomology associated to the short exact sequence
$$0\to\mathcal{H}^{-1}(\mathcal{E})\to 0\to\mathcal{H}^{-1}(\mathcal{Q})\to\mathcal{E}\to\mathcal{O}_S\to\mathcal{H}^0(\mathcal{Q})\to 0.$$
This shows $\mathcal{H}^{-1}(\mathcal{E}) = 0$ and thus $\mathcal{E}$ must be a sheaf. Since $\mathcal{O}_S\to\mathcal{H}^0(\mathcal{Q})\to 0$ is exact, we must have the kernel of this map is trivial or an idea sheaf of the form $\mathcal{I}_Z(-C)$, where C is an effective curve and Z is a zero-dimensional subvariety.

If the kernel is trivial, that is, $\mathcal{H}^0(\mathcal{Q})\cong\mathcal{O}_S$, then we would have $\mathcal{H}^{-1}(\mathcal{Q})\cong\mathcal{E}$. However, $\mathcal{H}^{-1}(\mathcal{Q})\in\mathscr{F},\mathcal{E}\in\mathscr{T}$ and $\mathscr{F}\cap\mathscr{T}=\{0\}$. Therefore, in this case, both $\mathcal{H}^{-1}(Q)$ and $\mathcal{E}$ would have to be zero.

This shows the kernel of $\mathcal{O}_S\to\mathcal{H}^0(\mathcal{Q})$ must be of the form $\mathcal{I}_Z(-C)$. Since $\mathcal{E}$ is an extension of torsion-free sheaves $\mathcal{I}_Z(-C)$ and $\mathcal{H}^{-1}(\mathcal{Q})$, it's also a torsion-free sheaf. □

In other words, in the case of Bridgeland stability, the subojects of $\mathcal{O}_S$ is a sheaf but may a priori have arbitrary high rank. Now let's fix a 3-slice $S_{\omega,G}$ and study which forms the wall $\mathscr{W}(\mathcal{E},\mathcal{O}_S)$ can take in $S_{\omega,G}$.

Let $\mathcal{E}$ be a torsion-free sheaf and set $\operatorname{ch}(\mathcal{E}) = (r, \operatorname{ch}_1(\mathcal{E}), c)$. We may write

$$\operatorname{ch}_1(\mathcal{E}) = d_h\omega + d_g G + \alpha,$$

where $\alpha \cdot \omega = \alpha \cdot G = 0, d_h = \operatorname{ch}_1(\mathcal{E}) \cdot \omega$ and $d_g = -\operatorname{ch}_1(\mathcal{E}) \cdot G$.

Let $\sigma_{t\omega, s\omega+uG}$ be a Bridgeland stability condition in the 3-slice $S_{\omega,G}$. For convenience we denote the heart of $\sigma_{t\omega,s\omega+uG}$ by $\mathscr{B}_{t,s,u}$, and denote the slope function of $\sigma_{t\omega,s\omega+uG}$ by $\beta_{t,s,u}$.

Lemma 1.3.3 ([AM16, Remark 4.6])**.** The short exact sequence $0 \to \mathcal{E} \to \mathcal{O}_S \to \mathcal{Q} \to 0$ in $\mathscr{B}_{t,s,u}$ if and only if

$$\mu_\omega(\overline{J}) < s < \mu_\omega(\underline{K}),$$

where

(1)
$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n = \mathcal{E}$$
is the Harder-Narasimhan filtration of $\mathcal{E}$ for slope stability and $\underline{K} = \mathcal{E}/\mathcal{E}_{n-1}$.

(2)
$$0 = \mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_m = \mathcal{H}^{-1}(\mathcal{Q})$$
is the Harder-Narasimhan filtration of $\mathcal{H}^{-1}(\mathcal{Q})$ for slope stability and $\overline{J} = \mathscr{F}_1$.

Moreover, we must have $\mu_\omega(\overline{J}) < \mu_\omega(\underline{K}) < 0$.

Proof. For $\mathcal{E}$ to be an object of $\mathscr{B}_{t,s,u}$, we must have $s < \mu_\omega(\underline{K})$ and for $\mathcal{H}^{-1}(\mathcal{Q})$ to be an objects of $\mathscr{B}_{t,s,u}$, we must have $s \geq \mu_\omega(\overline{J})$. The last condition $\mu_\omega(\underline{K}) < 0$ originates from the fact $\mu_\omega(\underline{K}) \leq \mu_\omega(\mathcal{E}) < 0$, where $\mu_\omega(\mathcal{E}) < 0$ since $\mathcal{E}$ is an extension of $\mathcal{I}_Z(-C)$ by $\mathcal{H}^{-1}(\mathcal{Q})$ for some effective curve C and zero-dimensional subvariety Z and $\mu_\omega(\mathcal{H}^{-1}(\mathcal{Q})) \leq \mu_\omega(\overline{J}) < 0$. □

Lemma 1.3.4 ([AM16, Lemma 4.7])**.**

(1) If $\mathscr{W}(\mathcal{E}, \mathcal{O}_S) \cap \prod_{u_0}$ intersects the line $s = \mu_\omega(\underline{K})$ for $t > 0$, then $\beta(\mathcal{E}_{n-1}) > \beta(\mathcal{E})$ at σ_0 with $\mathcal{E}_{n-1} \subseteq \mathcal{O}_S$ in $\mathscr{B}$
(2) If $\mathscr{W}(\mathcal{E}, \mathcal{O}_S) \cap \prod_{u_0}$ intersects the line $s = \mu_\omega(\overline{J})$ for $t > 0$, then $\beta(\mathcal{E}/\overline{J}) > \beta(\mathcal{E})$ at σ_0 with $\mathcal{E}/\overline{J} \subseteq \mathcal{O}_S$ in $\mathscr{B}$

Now let's $\mathcal{E} \subseteq \mathcal{O}_S$ be a suboject and consider the walls $\mathscr{W}(\mathcal{E}, \mathcal{O}_S)$. A direct compututation shows

$$Z_{t,s,u}(\mathcal{E}) = \left(-c + sd_h - ud_g - \frac{r}{2}(s^2 - u^2 - t^2)\right) + \sqrt{-1}(th_d - rst)$$

and the equation of the wall $\mathscr{W}(\mathcal{E}, \mathcal{O}_S)$ is

$$\frac{t}{2}\left(-d_h(s^2 + t^2 + u^2) + 2d_g su + 2cs)\right) = 0,$$

which is equivalent to

$$\left(-d_h(s^2 + t^2 + u^2) + 2d_g su + 2cs)\right) = 0$$

as $t \neq 0$.

At each fixed u, Maciocia showed in [Mac14] that all walls for $\mathcal{O}_S$ in the plane $\prod_u$ are nested semicircles centered on the s-aixs. Thus, given two subojects $\mathcal{E}_1$ and $\mathcal{E}_2$ and a fixed value u, we have $\mathscr{W}(\mathcal{E}_1, \mathcal{O}_S)$ and $\mathscr{W}(\mathcal{E}_2, \mathcal{O}_S)$ are both semicircles, with one of them inside the other one, unless they are equal.

Now we may think $S_{\omega,G}$ spaces as being extended to the $t = 0$ plane, and we study these quadrics by studying their intersection with $t = 0$:

$$-d_h(s^2 + u^2) + 2d_g su + 2cs = 0. \tag{1.1}$$

Since the walls are semicircles in $\prod_u$ for any u, knowing where the wall is at $t = 0$ would tell us where the wall is at any $t > 0$. The discriminant of (1.1) is

$$\Delta = 4(d_g^2 - d_h^2)$$

and (1.1) can be written as

$$-d_h(s + \frac{d_g}{d_h}u)^2 + \frac{\Delta}{4d_h}u^2 + 2cs = 0.$$

Note that $\mathcal{E} \subseteq \mathcal{O}_S$ in $\mathscr{B}_{t,s,u}$ implies $s < 0$ and by proof in Lemma 1.3.3 we have $d_h = \mu_\omega(\mathcal{E}) < 0$. Thus

(1) For $\Delta = 0$, the parabola case, it can only be a weakly destabilizing wall if $c \geq 0$.
(2) For $\Delta < 0$, the ellipse case, it can only be a weakly destabilizing wall if $c > 0$.
(3) For $\Delta > 0$, the hyperbola case, there are three cases given by $c = 0, c > 0$ and $c < 0$.

1.3.3. *Subojects of $\mathcal{O}_S$ of rank one.* Let $\sigma_{t\omega, s\omega+uG}$ be a Bridgeland stability in the 3-slice $S_{\omega,G}$. If a suboject $\mathcal{E} \subseteq \mathcal{O}_S$ in $\mathscr{B}_{t,s,u}$ is of rank one, then by Lemma 1.3.2 $\mathcal{E}$ must be equal to $\mathcal{I}_Z(-C)$ for some effective curve C and some zero-dimensional scheme Z with C or Z possibly 0.

Lemma 1.3.5. $\mathcal{I}_Z$ does not destabilize $\mathcal{O}_S$.

Proof. Let $i : Z \to X$ be a zero-dimensional scheme of length $\ell(Z)$. Then the Chern character of $\mathcal{I}_Z$ is

$$\mathrm{ch}(\mathcal{I}_Z) = (0, 0, -\ell(Z)),$$

and thus we have

$$\beta_{t,s,u}(\mathcal{I}_Z) = \frac{-2\ell(Z) + s^2 - u^2 - t^2}{-2st}.$$

On the other hand, we have

$$\beta_{t,s,u}(\mathcal{O}_S) = \frac{s^2 - u^2 - t^2}{-2st}.$$

Therefore, when $\mathcal{O}_S \in \mathscr{B}_{t,s,u}$, we have $s < 0$ and $\beta_{t,s,u}(\mathcal{I}_Z) < \beta_{t,s,u}(\mathcal{O}_S)$. This means that $\mathcal{I}_Z$ does not destabilize $\mathcal{O}_S$ whenever $\mathcal{O}_S \in \mathscr{B}_{t,s,u}$. □

Proposition 1.3.1 ([AM16, Proposition 5.1])**.** If $C^2 \geq 0$, then $\mathcal{I}_Z(-C)$ does not weakly destabilize $\mathcal{O}_S$.

Proof. Suppose $C = c_h\omega + c_G G + \alpha_C$ with $\alpha_C \cdot \omega = \alpha_C \cdot G = 0$. Then $C^2 = c_h^2 - c_g^2 + \alpha_C^2$, and $\alpha_C^2 \le 0$ by the Hodge index theorem. Theorefore, if $C^2 \ge 0$, then $c_h^2 - c_g^2 \ge 0$. Note that Chern character of $\mathcal{I}_Z(-C)$ is

$$\mathrm{ch}(\mathcal{I}_Z(-C)) = (1, -C, \frac{1}{2}C^2 - \ell(Z)).$$

Then the equation for the wall $\mathscr{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$ simplifies to

$$c_h(s^2 + t^2 + u^2) - 2c_g su + (c_h^2 - c_g^2)s + \alpha_C^2 s - 2\ell(Z)s = 0.$$

(1) If $c_h^2 - c_g^2 > 0$, then the wall at $t = 0$ is an ellipse going through $(0,0)$ and

$$P_W = \frac{C^2 - 2\ell(Z)}{c_g^2 - c_h^2}(c_h, c_g),$$

where these are the two points where the tangent line is vertical. Therefore, the s-value of any point on the ellipse is between 0 and the s-value of P_W, which is

$$\frac{C^2 - 2\ell(Z)}{c_g^2 - c_h^2}c_h = \frac{c_h^2 - c_g^2 + \alpha_C^2 - 2\ell(Z)}{c_g^2 - c_h^2}c_h = -c_h + \frac{\alpha_C^2 - 2\ell(Z)}{c_g^2 - c_h^2}c_h \ge -c_h.$$

But $\mathcal{I}_Z(C) \in \mathscr{B}_{t,s,u}$, we have that $s < -c_h$ and therefore $\mathcal{I}_Z(-C)$ cannot weakly destabilize $\mathcal{O}_S$.

(2) If $c_h^2 - c_g^2 = 0$, then $C^2 \ge 0$ implies $C^2 = 0$ and $\mathrm{ch}_2(\mathcal{I}_Z(-C))) = -\ell(Z) < 0$. As we list all possible weakly destabilizing wall for $\mathcal{O}_S$, this wall cannot be a weakly destabilizing wall.

□

1.3.4. *Bridgeland stability of $\mathcal{O}_S$ on surface without negative self-intersection curve.*

Theorem 1.3.1 ([AM16, Proposition 5.4])**.** Let S be a smooth projective complex surface and ω, B as before. If S does not contain any curves of negative self-intersection and $\sigma_{\omega,B}$ is a Bridgeland stability condition such that $\mathcal{O}_S \in \mathscr{B}_{\omega,B}$, then $\mathcal{O}_S$ is stable with respect to $\sigma_{\omega,B}$.

Proof. We prove the following statement by induction on the rank of $\mathcal{E}$: If $\mathcal{E} \subseteq \mathcal{O}_S$ is a proper subobject in $\mathscr{B}_{\omega,B}$ for some $\sigma_{\omega,B}$, then $\beta(\mathcal{E}) < \beta(\mathcal{O}_S)$ at $\sigma_{\omega,B}$.

If $\mathcal{E}$ has rank one, then it follows from Proposition 1.3.1. Assume now that $\mathcal{E}$ has rank $r > 1$ and that the result holds true for any proper subobject of rank less than r and any stability condition for which the object is indeed a subobject of $\mathcal{O}_S$. Choose G such that σ □

1.3.5. *Bridgeland stability of $\mathcal{O}_S$ on surface of Picard rank two.*

Theorem 1.3.2 ([AM16, Proposition 5.12])**.** Let S be a smooth projective complex surface of Picard rank two. Assume that the effective cone of S is generated by C_1 and C_2 such that $C_1 \cdot G > 0$ and C_1 is the only irreducible curve in S of negative self-intersection. Then $\mathcal{O}_S$ is only destabilized by $\mathcal{O}_S(-C_1)$.

2. Differential geometry aspects

2.1. **Backgrounds on deformed Hermitian-Yang-Mills metrics.**

Let X be a smooth projective complex variety and ω be an ample $\mathbb{R}$-divisor on X.

2.1.1. *Introduction.*

Definition 2.1.1. Let α be a real $(1,1)$-form on X. The *deformed Hermitian-Yang-Mills (dHYM) equation* seeks a function $\phi\colon X \to \mathbb{R}$ such that $\alpha_\phi = \alpha + \sqrt{-1}\partial\bar{\partial}\phi$, which satisfies

$$\operatorname{Im}(e^{-\sqrt{-1}\widehat{\theta}}(\omega + \sqrt{-1}\alpha_\phi)^n) = 0,$$

where

$$\int_X (\omega + \sqrt{-1}\alpha_\phi)^n \in \mathbb{R}_{>0}\, e^{\sqrt{-1}\widehat{\theta}}.$$

Remark 2.1.1. If we fix a point $p \in X$ and choose a holomorphic coordinate $\{z^i\}$ centered at p such that

$$\omega = \sqrt{-1}\sum_i \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i, \quad \alpha_\phi = \sqrt{-1}\sum_i \lambda_i \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i,$$

then the dHYM equation can be written as

$$\Theta_\omega(\alpha_\phi) = \widehat{\theta} \pmod{2\pi},$$

where $\Theta_\omega(\alpha_\phi) = \sum_i \arctan(\lambda_i)$ is called the *Lagrangian phase operator*.

Definition 2.1.2. Let $\mathcal{L} \to (X,\omega)$ be a line bundle. A Hermitian metric h on $\mathcal{L}$ is called a dHYM metric with respect to ω if the Chern curvature Θ_h satisfies

$$\operatorname{Im}\left(e^{-\sqrt{-1}\widehat{\theta}}\left(\omega - \frac{\Theta_h}{2\pi}\right)^n\right) = 0,$$

where

$$\int_X \left(\omega - \frac{\Theta_h}{2\pi}\right)^n \in \mathbb{R}_{>0}\, e^{\sqrt{-1}\widehat{\theta}}.$$

Remark 2.1.2.

(1) The dHYM metric on a line bundl $\mathcal{L}$ is a special case of the dHYM equation. If we choose real $(1,1)$-form $\alpha = \mathrm{ch}_1(\mathcal{L})$, then a solution of dHYM equation gives a dHYM metric on $\mathcal{L}$.
(2) Given a $\mathbb{R}$-divisor B on X, which is called a *B-field* in literature, a dHYM metric with respect to ω and B is a solution of dHYM equation defined by real $(1,1)$-class $\mathrm{ch}_1^B(\mathcal{L})$, where $\mathrm{ch}_1^B(\mathcal{L}) = e^{-B}\,\mathrm{ch}_1(\mathcal{L})$ is the *twisted Chern character*.
(3) The higher rank version of dHYM equation was proposed by Collins-Yau in [CY18, &8.1]. For a holomorphic vector bundle $\mathcal{E} \to (X,\omega)$, a Hermitian metric h is called a dHYM metric if the Chern curvature Θ_h satisfies

$$\operatorname{Im}\left(e^{-\sqrt{-1}\widehat{\theta}}\left(\omega \otimes \mathrm{id}_{\mathcal{E}} - \frac{\Theta_h}{2\pi}\right)^n\right) = 0,$$

where

$$\int_X \operatorname{tr}_h\left(\omega\otimes \mathrm{id}_{\mathcal{E}}-\frac{\Theta_h}{2\pi}\right)^n \in \mathbb{R}_{>0}\, e^{\sqrt{-1}\hat{\theta}}$$

and the imaginary part is defined[2] using the metric h. There are many fundamental results about dHYM metric for the line bundle, such as [JY17, CY18, CJY20], but the existence of the solution to the higher rank version is still in mystery.

2.1.2. *Twisted ampleness criterion.* In [CJY20], the authors proved a Nakai-Moishezon type criterion for the existence of dHYM metric of line bundles on Kähler surface[3] as follows, which is also called *twisted ampleness criterion* in [CLSY23].

Theorem 2.1.1 ([CJY20])**.** Let (X,ω) be a Kähler surface and $\mathcal{L}$ be a line bundle on X such that $\omega\cdot \mathrm{ch}_1(\mathcal{L})>0$. Then $\mathcal{L}$ admits a dHYM metric (with respect to ω) if and only if for every curve $C\subseteq X$ we have

$$\operatorname{Im}\left(\frac{Z_C(\mathcal{L})}{Z_X(\mathcal{L})}\right)>0,$$

where $Z_C(\mathcal{L})=-\int_C e^{-\sqrt{-1}\omega}\,\mathrm{ch}(\mathcal{L})$ and $Z_X(\mathcal{L})=-\int_X e^{-\sqrt{-1}\omega}\,\mathrm{ch}(\mathcal{L})$.

In [FYZ23], the authors showed the solution to dHYM equation on a compact Kähler surface (X,ω) always exists on the complement of a finite number of curves of negative self-intersection. In particular, the twisted ampleness criterion is satisfied automatically if there is no negative self-intersection curve on X. It motivates us to consider

Question 2.1.1. *Whether it suffices to test negative self-intersection curves in twisted ampleness criterion or not.*

In some cases, Question 2.1.1 can be checked directly, such as Hirzebruch surface, which also serves as an important example later. Let's give a brief review of cone structures and intersection form on Hirzebruch surface.

Proposition 2.1.1. Let $X=\mathscr{H}_r$ be the Hirzebruch surface and $\{D_1,D_2,D_3,D_4\}$ be the generators of torus-invariant divisors on X. Then

(1) ([CLS11, Example 4.1.8]) The Picard group is generated by $\{D_1,D_2,D_3,D_4\}$ with relations

$$0\sim \mathrm{div}(\chi^{e_1})=-D_1+D_3$$
$$0\sim \mathrm{div}(\chi^{e_2})=rD_1+D_2-D_4.$$

[2]To be precisely, via the Hermitian metric, or rather the induced metric on $\mathrm{End}(\mathcal{E})$, one can define the adjoint of any section of $\mathrm{End}(\mathcal{E})$, and then give a decomposition of any such a section into self adjoint and anti-self adjoint part. For example, given a complex matrix A, one can decompose it as

$$A=\frac{1}{2}(A+A^*)+\sqrt{-1}\times\frac{1}{2\sqrt{-1}}(A-A^*),$$

where $\frac{1}{2\sqrt{-1}}(A-A^*)$ is called the imaginary part.

[3]The higher dimensional case was proved in [CLT24].

(2) ([CLS11, Proposition 4.3.3]) The effective cone of X is given by

$$\mathrm{Eff}(X) = \{aD_1 + bD_2 \mid a, b \geq 0\}.$$

(3) ([CLS11, Example 6.1.17]) The ample cone of X is given by

$$\mathrm{Amp}(X) = \{\alpha D_1 + \beta D_4 \mid \alpha, \beta > 0\}.$$

(4) ([CLS11, Example 6.3.6]) The intersection matrix of D_1 and D_2 is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix}.$$

Proposition 2.1.2. Let $X = \mathscr{H}_r$ be Hirzebruch surface. Then for the twisted ampleness criterion on X, it suffices to test the only negative self-intersection curve.

Proof. Let $\omega = \alpha D_1 + \beta D_4$ be an ample $\mathbb{R}$-divisor on X and $\mathcal{L} = kD_3 + \ell D_4$ be a line bundle such that $\omega \cdot \mathrm{ch}_1(\mathcal{L}) > 0$. For curve $C \subseteq X$, the twisted ampleness criterion for C can be rewritten as

$$(C \cdot \mathrm{ch}_1(\mathcal{L}))(\omega \cdot \mathrm{ch}_1(\mathcal{L})) > \left(\mathrm{ch}_2(\mathcal{L}) - \frac{1}{2}\omega^2\right)(C \cdot \omega). \tag{2.1}$$

Take $C = D_1$, equation (2.1) gives

$$\ell(\alpha\ell + \beta k + r\beta\ell) > \frac{1}{2}(2k\ell + r\ell^2 - 2\alpha\beta - r\beta^2)\beta,$$

which is equivalent to

$$\alpha\ell^2 + \alpha\beta^2 + \frac{1}{2}(r\beta\ell^2 + r\beta^3) > 0. \tag{2.2}$$

It's clear that equation (2.2) holds for arbitrary $k, \ell \in \mathbb{Z}$ since $\alpha, \beta > 0$. This completes the proof since the twisted ampleness criterion is linear with respect to the intersection with C. □

3. Relations between algebraic geometry and differential geometry aspects

3.1. **History and conjectures.** It's a fundamental principle which has guided much of the research in complex geometry since the late 20-th century: stable objects in algebraic geometry should correspond to extremal objects in differential geometry. This philosophy can be traced back to study of the relations between the slope stability and the existence of Hermitian-Einstein metric. The curve case was established by Narasimhan-Seshadri.

Theorem 3.1.1 ([NS65])**.** A holomorphic vector bundle $\mathcal{E}$ on a compact Riemann surface is stable if and only if there is an irreducible Hermitian-Einstein metric on $\mathcal{E}$.

The further work on this topic is summarized as follows:

(1) Donaldson gave a new proof of Narasimhan-Seshadri's result in [Don83], and then he proved the surface case in [Don85].
(2) Uhlenbeck-Yau generalized this to compact Kähler manifold in [UY86].
(3) The non-Kähler case was proved in [LY87].
(4) For Higgs bundles, Simpson showed that every stable Higgs bundle has a Hermitian-Yang-Mills metric in [Sim88].
(5) Yau conjectured that the existence of Kähler-Einstein metric on Fano manifold should be equivalent to some algebro-geometric stability conditions, which was solved in [CDS15a], [CDS15b] and [CDS15c].

On the other hand, going back to the work of Douglas, and Thomas-Yau ([TY02]), it has long been conjectured that the existence of special Lagrangians (or solutions of dHYM equation) is equivalent to a purely algebraic notion of stability. This proposal is based on the idea that, in certain limits, a special Lagrangian should be mirror to a holomorphic bundle $\mathcal{E}$ with Hermitian-Yang-Mills connection, and by the Donaldson-Uhlenbeck-Yau Theorem, this is equivalent to $\mathcal{E}$ being slope stable.

Lemma 3.1.1. Let $\mathcal{L}\to(X,\omega)$ be a line bundle over a compact Kähler n-manifold. In the large volume region $k\gg 0$, the leading order condition[4] for $\mathcal{L}$ to admit a dHYM metric is given by the Hermitian-Einstein equation.

On the other hand, the large volume limit of Bridgeland stability is related to the slope stability.

Lemma 3.1.2 ([MS17, Lemma 6.18])**.** Let X be a smooth projective complex variety and $\omega, B\in N^1(X)$ with ω is ample. Let $\mathscr{B}_{\omega,B}$ be the tilt heart given by the torsion-free pair $\langle\mathscr{T}_{\omega,B}[1],\mathscr{T}_{\omega,B}\rangle$. If $\mathcal{E}\in\mathscr{B}_{\omega,B}$ is $\sigma_{\alpha\omega,B}$-semistable for all $\alpha\gg 0$, then it satisfies one of the following conditions:

[4]To be precise, the dHYM equation for $k\omega$ takes the form

$$Ck^{n-1}\omega^n+O(k^{n-3})=nk^{n-1}\omega^{n-1}\wedge c_1(\mathcal{L})+O(k^{n-3})$$

for some topological constant C determined by θ.

(1) $\mathcal{H}^{-1}(\mathcal{E})=0$ and $\mathcal{H}^0(\mathcal{E})$ is a $\mu_{\omega,B}$-semistable torsion-free sheaf.
(2) $\mathcal{H}^{-1}(\mathcal{E})=0$ and $\mathcal{H}^0(\mathcal{E})$ is a torsion sheaf.
(3) $\mathcal{H}^{-1}(\mathcal{E})$ is a $\mu_{\omega,B}$-semistable torsion-free sheaf and $\mathcal{H}^0(\mathcal{E})$ is either 0 or a torsion sheaf supported in dimension zero.

Proof. Let $\nu_{\omega,B}$ denote the slope function given by the Bridgeland stability condition $\sigma_{\omega,B}$. Since the stability won't change with scalling a constant, one can compute $\sigma_{\alpha\omega,B}$-stability with $2\nu_{\alpha\omega,B}/\alpha$ instead of $\nu_{\alpha\omega,B}$. It's convenient in the present argument because

$$\lim_{\alpha\to\infty}\frac{2\nu_{\alpha\omega,B}}{\alpha}(\mathcal{E})=-\mu^{-1}_{\omega,B}(\mathcal{E}).$$

By definition of the tilt heart $\mathscr{B}_{\omega,B}$, the object $\mathcal{E}$ is an extension $0\to\mathcal{F}[1]\to\mathcal{E}\to\mathcal{T}\to 0$, where $\mathcal{F}\in\mathscr{F}_{\omega,B}$ and $\mathcal{T}\in\mathscr{T}_{\omega,B}$. Moreover, $\mathscr{B}_{\omega,B}=\mathscr{B}_{\alpha\omega,B}$ for all $\alpha>0$.

Suppose $\omega\cdot\mathrm{ch}_1^B(\mathcal{E})=0$. Then both $\omega\cdot\mathrm{ch}_1^B(\mathcal{F})=0$ and $\omega\cdot\mathrm{ch}_1^B(\mathcal{T})=0$. By definition of $\mathscr{F}_{\omega,B}$ and $\mathscr{T}_{\omega,B}$ this means $\mathcal{T}$ is 0 or has be supported in dimension 0 and $\mathcal{F}$ is 0 or a $\mu_{\omega,B}$-semistable torsion free sheaf with $\mu_{\omega,B}(\mathcal{F})=0$. Therefore, for the rest of the proof we can assume $\omega\cdot\mathrm{ch}_1^B(\mathcal{E})>0$.

Suppose that $\mathrm{ch}_0^B(\mathcal{E})\geq 0$. Then the inequality $\omega\cdot\mathrm{ch}_1^B(\mathcal{E})>0$ implies $-\mu^{-1}_{\omega,B}(\mathcal{E})<0$. By definition we have $-\mu^{-1}_{\omega,B}(\mathcal{F}[1])\geq 0$, and since $\mathcal{E}$ is $\sigma_{\alpha\omega,B}$-semistable for $\alpha\gg 0$, we get $\mathcal{F}=0$ and $\mathcal{E}\in\mathscr{T}_{\omega,B}$ is a sheaf. If $\mathcal{E}$ is torsion, we are in case (2). Assume $\mathcal{E}$ is neither torsion nor slope semistable. Then by definition of $\mathscr{T}_{\omega,B}$ there is an exact sequence

$$0\to\mathcal{A}\to\mathcal{E}\to\mathcal{B}\to 0$$

in $\mathscr{T}_{\omega,B}$ such that $\mu_{\omega,B}(\mathcal{A})>\mu_{\omega,B}(\mathcal{E})>0$. But then $-\mu^{-1}_{\omega,B}(\mathcal{A})>-\mu^{-1}_{\omega,B}(\mathcal{E})$ contradicts the fact that $\mathcal{E}$ is $\sigma_{\alpha\omega,B}$-semistable for $\alpha\gg 0$.

Suppose that $\mathrm{ch}_0^B(\mathcal{E})<0$. If $\omega\cdot\mathrm{ch}_1^B(\mathcal{T})>0$, then $-\mu^{-1}_{\omega,B}(\mathcal{T})<0$ and the assumption $\mathrm{ch}_0^B(\mathcal{E})<0$ implies $-\mu^{-1}_{\omega,B}(\mathcal{E})>0$. Then the fact that $\mathcal{E}$ is $\sigma_{\alpha\omega,B}$-semistable for $\alpha\gg 0$ gives a contradiction.

Now we assume $\omega\cdot\mathrm{ch}_1^B(\mathcal{T})=0$. Then $\mathcal{T}\in\mathscr{T}_{\omega,B}$ implies $\mathrm{ch}_0^B(\mathcal{T})=0$ and $\mu_{\omega,B}(\mathcal{E})=\mu_{\omega,B}(\mathcal{F})$. It remains to show $\mathcal{F}$ is $\mu_{\omega,B}$-semistable. If not, there is an exact sequence

$$0\to\mathcal{A}\to\mathcal{F}\to\mathcal{B}\to 0$$

in $\mathscr{F}_{\omega,B}$ such that $\mu_{\omega,B}(\mathcal{A})>\mu_{\omega,B}(\mathcal{F})$. Therefore, there is an injective map $\mathcal{A}[1]\hookrightarrow\mathcal{E}$ in $\mathscr{B}_{\omega,B}$ such that $-\mu^{-1}_{\omega,B}(\mathcal{A}[1])>-\mu^{-1}_{\omega,B}(\mathcal{F})=-\mu^{-1}_{\omega,B}(\mathcal{E})$ in contradiction to the fact that $\mathcal{E}$ is $\sigma_{\alpha\omega,B}$-semistable for $\alpha\gg 0$. □

The present version of this folklore conjecture is

Conjecture 3.1.1. A line bundle $\mathcal{L}$ admits a dHYM metric if and only if it is stable in the sense of Bridgeland as an object in $D^b(X)$.

3.2. **Known results.** Until now, even for surface case, the relations between Bridgeland stability and the existence of dHYM metric are still in mystery in general. Before we summarize some known results, we firstly set up our settings: Let X be a smooth projective complex surface with ample $\mathbb{R}$-divisor ω and B be a $\mathbb{R}$-divisor on X. Let $\sigma_{\omega,B} = (Z_{\omega,B}, \mathscr{B}_{\omega,B})$ be the Bridgeland stability with the central charge $Z_{\omega,B}$ given by

$$Z_{\omega,B} = -\int_X e^{-\sqrt{-1}\omega} \operatorname{ch}^B$$

and $\mathscr{B}_{\omega,B} = \langle \mathscr{F}_{\omega,B}[1], \mathscr{T}_{\omega,B}\rangle$ be *tilt heart* given by the following torsion pair

$$\mathscr{T}_{\omega,B} = \{\mathcal{E} \in \operatorname{Coh}(X) \mid \mu_{\omega,B,\min}(\mathcal{E}) > 0\}$$
$$\mathscr{F}_{\omega,B} = \{\mathcal{E} \in \operatorname{Coh}(X) \mid \mu_{\omega,B,\max}(\mathcal{E}) \leq 0\},$$

where $\mu_{\omega,B}(\mathcal{E}) = \operatorname{ch}_1^B(\mathcal{E})/\operatorname{ch}_0(\mathcal{E})$.

3.2.1. *Surface without negative self-intersection curve.* Let X be a smooth projective complex surface without negative self-intersection curve. For any ample $\mathbb{R}$-divisor and $\mathbb{R}$-divisor B on X, by results in [FYZ23], every line bundle on X admits a dHYM metric with respect to ω and B. On the other hand, by results in [AM16] shows that any line bundle $\mathcal{L}$ is also Bridgeland stable with respect to $\sigma_{\omega,B}$. In particular, the Conjecture 3.1.1 holds trivially.

3.2.2. *Counter-examples with zero B-field.* The main techniques for counter-example in [CS22] are twisted ampleness criterion and results in [AM16].

Proposition 3.2.1. Let $X = \mathscr{H}_1 = \operatorname{Bl}_p \mathbb{P}^2$ and ω be an ample $\mathbb{R}$-divisor on X. Let $\mathcal{L}$ be a line bundle on X with $\omega \cdot \operatorname{ch}_1(\mathcal{L}) > 0$. If $\mathcal{L}$ admits a dHYM metric with respect to ω, then $\mathcal{L}$ is Bridgeland stable at $\sigma_{\omega,0}$. The converse statement is not true.

Proof. A line bundle $\mathcal{L} \in \mathscr{B}_{\omega,0}$ if and only if $\omega \cdot \operatorname{ch}_1(\mathcal{L}) > 0$, and by Theorem 1.3.2 it's Bridgeland stable at $\sigma_{\omega,0}$ if and only if

$$\beta(\mathcal{L}(-C)) < \beta(\mathcal{L}), \tag{3.1}$$

where ρ is the slope function given by $\sigma_{\omega,0}$. A direct computation shows that (3.1) is equivalent to

$$\left(C \cdot \operatorname{ch}_1(\mathcal{L}) - \frac{1}{2}C^2\right)(\omega \cdot \operatorname{ch}_1(\mathcal{L})) > \left(\operatorname{ch}_2(\mathcal{L}) - \frac{1}{2}\omega^2\right)(C \cdot \omega), \tag{3.2}$$

where $C \subseteq X$ is the only curve which has negative self-intersection.

On the other hand, the twisted ampleness criterion (Theorem 2.1.1) shows that $\mathcal{L}$ admits a solution to dHYM equation if and only if

$$(C \cdot \operatorname{ch}_1(\mathcal{L}))(\omega \cdot \operatorname{ch}_1(\mathcal{L})) > \left(\operatorname{ch}_2(\mathcal{L}) - \frac{1}{2}\omega^2\right)(C \cdot \omega), \tag{3.3}$$

for all curves $C \subseteq X$. As a consequence, (3.3) implies (3.2), that is, every line bundle admitting a dHYM metric is Bridgeland stable with respect to $\sigma_{\omega,0}$.

Conversely, let $\omega = \frac{1}{\sqrt{3}}(D_1 + D_4)$ and $\mathcal{L} = \mathcal{O}_X(2D_4)$. A direct computation shows $\mathcal{L}$ is Bridgeland stable with respect to $\sigma_{\omega,0}$, but does not admit a dHYM metric with respect to ω. □

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