# Intersection Theory of Toric varieties

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# Chapter 1

# Chow Groups

We may introduce the general theory of intersection. Our main reference are [2] and Stacks Project. All schemes are of finite type over a given field and all morphisms are over this field, unless specified otherwise.

## 1.1 Algebraic Cycles

**Definition 1.1.1 (algebraic cycle)** A *$k$-cycle* of a scheme $X$ is an element of the free abelian group $Z_k(X)$ generated by the closed integral subschemes of dimension $k$.

We have the ($\mathbb{N}$-)graded group $Z_\bullet(X) = \bigoplus_{k\geq 0} Z_k(X)$, and generally a *cycle* refers an element in $Z_\bullet(X)$. We also have the concept of *$k$-cocycle*, which is an element of $Z^{n-k}(X)$ if $X$ has pure dimension $n$. An 1-cocycle is nothing else but a Weil divisor when $X$ is integral.

For a cycle

$$\alpha = \sum n_Z Z,$$

we can define its *support* by $\mathrm{supp}(\alpha) = \bigcup_{n_Z \neq 0} Z$.

**Definition 1.1.2 (effective cycle)** An *effective cycle* is a cycle of non-negative coeffecients. Any effective cycle is associated to a closed subscheme.

Let $Z$ by a closed subscheme of $X$. We define the cycle associated to $Z$ by

$$\sum_{\dim \mathscr{O}_{Z,\xi}=0} \operatorname{length}(\mathscr{O}_{Z,\xi}) \cdot \overline{\{\xi\}}.$$

It is a cycle of support $Z$. Note that $\overline{\{\xi\}}$ runs through (finitely many) irreducible components of $Z$, and $\mathscr{O}_{Z,\xi}$ is an artinian local ring hence has finite length.

The pushforward of cycles are well-defined, makes $Z_\bullet$ a functor.

**Definition 1.1.3 (pushforward)** For a morphism $f\colon X \to Y$ and an integral closed subscheme $V \subset X$ of dimension $k$, $W := \overline{f(V)}$ is an integral closed subscheme of $Y$. If $\dim V = \dim W$, then $K(V)$ is a finite extension of $K(W)$ (see 02NX). We define

$$f_*V = \begin{cases} [K(V) : K(W)] \cdot W, & \dim V = \dim W, \\ 0, & \text{otherwise}, \end{cases}$$

which extends linearly to a homomorphism

$$f_*\colon Z_k(X) \to Z_k(Y).$$

For example, if $i\colon V \to X$ is an closed immersion, then the pushforward $i_*$ identifies $Z_k(V)$ as a subgroup of $Z_k(X)$ in an obvious way.

**Definition 1.1.4 (exterier product)** We have the obvious *exterier product* or *Künneth map*

$$\begin{aligned} Z_\bullet(X) \otimes Z_\bullet(Y) &\to Z_\bullet(X \times Y) \\ (U, V) &\mapsto U \times V \end{aligned}$$

## 1.2 Rational Equivalence

Algebraic cycles generalize the concept of Weil divisors, and rational equivalence is the analog of linear equivalence.

**Definition 1.2.1 (rational equivalence)** Suppose $V$ is a $(k+1)$-dimensional integral closed subscheme $V$ of $X$. For a nonzero rational function $f \in K(V)^*$, we can define its *principal divisor* $\mathrm{div}_V(f)$ by

$$\sum_{\dim \mathscr{O}_{V,z}=1} \mathrm{ord}_z(f) \cdot \overline{\{z\}},$$

as in 0BE3. We have the subgroup $R_k(X) \subset Z_k(X)$ generated by principal divisors of every $(k+1)$-dimensional integral closed subschemes of $X$, and the graded subgroup $R_\bullet(X) = \bigoplus_k R_k(X)$.

Any tow cycles $\alpha, \beta$ with $\alpha - \beta \in R_\bullet(X)$ is called *rationally equivalent*, denoted by $\alpha \sim \beta$. The quotient groups

$$A_\bullet(X) = Z_\bullet(X)/R_\bullet(X),$$

is called the *Chow group* of $X$.

*Remark.* The Chow group of a scheme $X$ is the analog of the singular homology of a topological space.

If $X$ is an integral scheme of finite type over $k$, with $\dim X = n$, then there are some easy observations that

- $A_{n-1}(X) = \mathrm{Cl}(X)$ is just the Weil divisor class group,
- $A_n(X) = Z_n(X)$ is just the free abelian generated by $X$ itself,
- $A_k(X) = Z_n(X) = 0$ for $k > n$.

However, it is difficult to determine $A_k(X)$ for $k < n$, even if $X$ is affine or $k = 0$.

**Proposition 1.2.2** The exterier product of two cycles rationally equivalent to zero is also rationally equivalent to zero. So we have exterier product

$$A_\bullet(X) \otimes A_\bullet(Y) \to A_\bullet(X \times Y),$$

of Chow groups.

## 1.3 Proper Pushforward

Suppose $f\colon X \to Y$ is a proper morphism, the pushforward $f_*$ will behave well.

**Proposition 1.3.1** If $V$ is an integral closed subscheme of $X$ and $h$ is a nonzero rational function on $V$, then

$$f_*\mathrm{div}_V(h) = \begin{cases} \mathrm{div}\left(\mathrm{Norm}_{K(V)/K(f(V))}(h)\right), & \dim V = \dim f(V), \\ 0, & \dim V > \dim f(V). \end{cases}$$

*Proof.* This is not easy, see 02RT. □

This proposition shows that $f$ can induce the morphism

$$f_*\colon A_\bullet(X) \to A_\bullet(Y),$$

of Chow groups, which is the so-called *proper pushforward*.

*Remark.* There are other advantages of proper morphism. For example, since $f$ (in fact universally) closed, for any integral closed subscheme $V \subset X$, $f(V)$ is already closed and $\dim f(V) \leq \dim V$.

*Remark.* A closed immersion $i\colon V \to X$ is always proper, so it induces a morphism $i_*\colon A_k(V) \to A_k(X)$. This is not in general injective, even if $Z_k(V)$ is a subgroup of $Z_k(X)$.

**Definition 1.3.2 (degree of 0-cycles)** Consider the structure morphism $X \to \operatorname{Spec} k$, we have the induced morphism

$$Z_0(X) \to Z_0(\operatorname{Spec} k) = \mathbb{Z},$$

which defines the *degree* of 0-cycles. When $X$ is proper over $k$, the degree morphism

$$\deg\colon A_0(X) \to \mathbb{Z},$$

is well-defined. If $\alpha = n_1 x_1 + \cdots + n_r x_r$, then we have explicitly

$$\deg \alpha = \sum_{i=1}^{r} n_i [\kappa(x_i)/k],$$

where $\kappa(x_i)$ is the residue field of $\mathscr{O}_{X,x_i}$.

*Remark.* When $X$ is a smooth complete variety of dimension $n$, then we can define

$$\int_X : A^\bullet(X) \to \mathbb{Z},$$

which sends a cocycle to the degree of its component of degree $n$. The integral symbol comes from the Poincaré duality of a compact manifold.

## 1.4 Flat Pullback

There are various ways to define the pullback of a cycle. We first introduce the flat pullback.

**Definition 1.4.1 (pullback)** For a morphism $f\colon X \to Y$ and an integral closed subscheme $W \subset Y$, we can consider the scheme theoretic inverse image

$$f^*V := W \times_Y X,$$

which is a closed subscheme of $X$ with underlying topological space $f^{-1}(V)$ (see exercise II.3.11(a) in [1, pp92]). This extends linearly to a map

$$f^*\colon Z_\bullet(Y) \to Z_\bullet(X).$$

When $f\colon X \to Y$ is flat which has relative dimension, this pullback $f^*$ will behave well.

**Proposition 1.4.2** If $f$ is a flat morphism of relative dimension $r$, then

1. If $W$ is a $k$-dimensional integral closed subscheme of $Y$, then $f^{-1}W$ has pure dimension $k+r$.

2. If $\alpha \sim \beta$, then $f^*\alpha \sim f^*\beta$.

*Proof.* See 02R8, 02S1. ☐

This proposition shows that $f$ can induce the morphism

$$f^*\colon A_k(Y) \to A_{k+r}(X),$$

of groups, which is the so-called *flat pullback*.

**Example 1.4.3** If $f\colon X \to Y$ is finite locally free of degree $d$ (e.g. nonconstant morphism of projective curves), then $f$ is proper and flat of dimension 0, so we have morphism of graded rings

$$f_*\colon A_\bullet(X) \to A_\bullet(Y),$$
$$f^*\colon A_\bullet(Y) \to A_\bullet(X),$$

and we have $f_*f^*\alpha = d\alpha$. See 02RH.

**Example 1.4.4 (vector bundle)** If $p\colon E \to X$ is a vector bundle of rank $r$, then it is flat of relative dimension $r$. The induced morhism

$$p^*\colon A_k(X) \to A_{k+r}(E),$$

is an isomorphism. See [2, pp64].

**Example 1.4.5 (localization sequence)** If $j\colon U \to X$ is an open immersion, then it is flat of relative dimension 0. We have the exact sequence

$$A_k(Z) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0,$$

where $i\colon Z \to X$ is a closed immersion of reduced closed subscheme $Z = X - U$. This is called *localization sequence*, which is often used to compute Chow groups. See [2, pp21].

# Chapter 2

# Intersection Products

## 2.1 Intersection with Cartier Divisors

We may introduce some facts about Cartier divisors, one can see e.g. section II.6 of [1] for details. Let $X$ be an integral scheme, a *Cartier divisor* refers a global section of the quotient sheaf $\mathscr{K}^*/\mathscr{O}_X^*$. Cartier divisors form a group $\mathrm{CaDiv}(X)$, and has a quotient $\mathrm{CaCl}(X)$ modulo linear equivalence.

Each Cartier divisor $D$ on $X$ can associate to a Weil divisor, and this produces a homomorphism

$$\mathrm{CaCl}(X) \to \mathrm{Cl}(X).$$

This homomorphism is injective if $X$ is normal, and is bijective if $X$ is smooth.

Each Cartier divisor $D$ on $X$ can also associate to an invertible sheaf $\mathscr{O}_X(D)$, and this produces an isomomorphism

$$\mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Pic}(X).$$

**Definition 2.1.1** For a Cartier divisor $D$ on $X$ and a $k$-dimensional irreducible closed variety $V \subset X$, the restriction (or pullback) $\mathscr{O}_X(D)|_V$ in an invertible sheaf on $V$. So there is a unique Cartier divisor class $D.V$ on $V$ such that $\mathscr{O}_X(D)|_V \simeq \mathscr{O}_V(D.V)$. The map $(D, V) \mapsto D.V$ extends to a morphism

$$\mathrm{CaDiv}(X) \times Z_k(X) \to A_{k-1}(X),$$

which is the *intersection* of Cartier divisors and $k$-cycles.

*Remark.* If $V \not\subset \mathrm{supp}(D)$, then the restriction $D|_V$ is well-defined by restricting the local equations. For general case, one can also easily see that $D.V$ is well-defined in $A_{k-1}(V \cap \mathrm{supp}(D))$.

We may introduce some facts about intersection of Cartier divisors and cycles, see [2] for details.

**Proposition 2.1.2**

- If $D_1 \sim D_2$, $\alpha_1 \sim \alpha_2$, then $D_1.\alpha_1 = D_2.\alpha_2$.
- For Cartier divisors $D_1, D_2$, identifying $D_2$ as an 1-cocycle then we have $D_1.D_2 = D_2.D_1$ in $A^2(X)$.

Thus the intersection can induce morphism

$$\mathrm{Pic}(X) \times A_k(X) \to A_{k-1}(X),$$

and a symmetric bilinear map

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to A^{k-2}(X).$$

Sometimes we also treat with $\mathbb{Q}$-Cartier divisors, i.e., elements in $\mathrm{Div}(X)_{\mathbb{Q}}$ with some multiple Cartier. In this case we can define the intersection of a $\mathbb{Q}$-Cartier and a $\mathbb{Q}$-cycle, which is also a $\mathbb{Q}$-cycle, and we have the bilinear map

$$\mathrm{Pic}(X)_{\mathbb{Q}} \times A_k(X)_{\mathbb{Q}} \to A_{k-1}(X)_{\mathbb{Q}}.$$

## 2.2 Geometric Intersection

Let $X$ be an irreducible variety over an algebraically closed field (e.g. $\mathbb{C}$), and $Y_1, \cdots, Y_r$ be irreducible closed subvarieties.

**Definition 2.2.1 (proper intersection)** We say $Y_1, \cdots, Y_r$ *intersect properly* if

$$\mathrm{codim}\left(\bigcap_{i=1}^{r} Y_i\right) = \sum_{i=1}^{r} \mathrm{codim}(Y_i).$$

Here the codim is the codimension of a closed subvariety in $X$.

**Definition 2.2.2 (transversal intersection)** If $X, Y_i$ are all smooth, we say $Y_1, \cdots, Y_r$ *intersect transversally* if for each $p \in \bigcap_{i=1}^r Y_i$ we have

$$\operatorname{codim}\left(\bigcap_{i=1}^{r} T_p Y_i\right) = \sum_{i=1}^{r} \operatorname{codim}(T_p Y_i).$$

Here the codim is the codimension of a linear subspace in $T_p X$.

**Definition 2.2.3 (Serre's intersection multiplicity)** If $Y, Z$ intersect properly, let $W$ be an irreducible component of $Y \cap Z$ with regular generic point. We define the *intersection multiplicity* of $Y, Z$ at $W$ to be

$$i(Y, Z; W) = \sum_i (-1)^i \text{length}\, \operatorname{Tor}_i^A(A/I, A/J),$$

where $A = \mathscr{O}_{X,W}$ is the local ring of $X$ at the generic point of $W$, and $I, J$ be the ideals of $Y, Z$ respectively. Note that $A$ is a regular local ring hence has finite global dimension.

*Remark.* If $Y$ intersects $Z$ transversally, then $i(Y, Z; W) = 1$ for each component $W$.

## 2.3 Chow Rings

Now let $X$ be an irreducible smooth variety over an algebraically closed field $k$. Fulton constructed an *intersection product* on $A^\bullet(X)$ such that $A^\bullet(X)$ is a commutative associative graded ring, and such ring $A^\bullet(X)$ is called the *Chow ring* of $X$. The intersection product satisfis some expected properties.

**Proposition 2.3.1** In the Chow ring $A^\bullet(X)$ of $X$, we have

- If $Y, Z$ are irreducible smooth closed subvarieties of $X$ intersects properly, then
$$[Y] \cdot [Z] = \sum_W i(Y, Z; W)[W],$$

where the sum runs over all components of $Y \cap Z$.

- If $V$ is an irreducible closed subvariety and $D$ is an effective Cartier divisor, then $[D] \cdot [V]$ is just the intersection $D.V$ defined before.
- $A^\bullet$ defines a contravariant functor from the category of irreducible smooth varieties to the category of commutative associative graded rings, and the pullback $f^*$ is just the flat pullback when $f$ is flat (of relative dimension 0).

*Remark.* By *Chow's moving lemma*, if $X$ is moreover quasi-projective, the intersection product is uniquely determined by above proposition.

It is difficult to compute Chow rings in general, here are some selected facts.

**Example 2.3.2** $A^\bullet(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n]$ with $\deg[\mathbb{A}^n] = n$.

**Example 2.3.3** By the classical Bézout theorem, one has

$$A^\bullet(\mathbb{P}^2) = \mathbb{Z}[h]/(h^3),$$

is the truncated polynomial ring, where $h$ is the class of any line.

*Remark.* For some special singular varieties , the *rational Chow ring* $A^\bullet_{\mathbb{Q}}$ can be defined. In toric world, we are intersted in rational Chow ring of a simplicial toric variety.

# Chapter 3

# Riemann-Roch Theorem

See appendix A of [1] for details. Let $X$ be a smooth variety (over $\mathbb{C}$).

## 3.1 Chern Class

**Definition 3.1.1 (Chern class)** For a locally free sheaf $\mathscr{E}$ on $X$, there is a projective bundle $\pi\colon \mathbb{P}(\mathscr{E}) \to X$. The invertible sheaf $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ corresponds to an 1-cocycle $\xi \in A^1(\mathbb{P}(\mathscr{E}))$. One defines the *Chern class* $c(\mathscr{E}) \in A^\bullet(X)$ by

- $c_0(\mathscr{E}) = 1$.
- $c_i(\mathscr{E}) = 0$ for $i > r$.
- $\sum_{i=0}^{r}(-1)^i(\pi^* c_i(\mathscr{E})) \cdot \xi^{r-i}$.
- $c(\mathscr{E}) = \sum_{i\geq 0} c_i(\mathscr{E})$.

**Proposition 3.1.2** Chern classes of locally free sheaves have the following properties.

- $c(\mathscr{O}_X(D)) = 1 + D$ for Cartier divisor $D \in A^1(X)$.
- If $f\colon X \to Y$ is a morphism of smooth varieties and $\mathscr{E}$ is a locally free sheaf on $Y$, then $c(f^*\mathscr{E}) = f^* c(\mathscr{E})$.

- If $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ is an exact sequence of locally free sheaves, then $c(\mathscr{F}) = c(\mathscr{E}) \cdot c(\mathscr{G})$.

**Lemma 3.1.3 (splitting principle)** For a locally free sheaf $\mathscr{E}$ of rank $r$ on $X$, there is a morphism $f\colon X' \to X$ such that $f^*\colon A^\bullet(X) \to A^\bullet(X')$ is injective and there is a filtration

$$0 = \mathscr{F}_0 \subset \cdots \subset \mathscr{F}_r = f^*\mathscr{E},$$

such that $\mathscr{F}_i/\mathscr{F}_{i-1}$ is an invertible sheaf for each $i$.

The splitting principle shows that the Chern class can be uniquely defined by proposition 3.1.2. Moreover, if $\mathscr{E}$ is locally free of rank $r$, then there are $\xi_i \in A^1(X)$ such that

$$c(\mathscr{E}) = \prod_{i=1}^{r}(1 + \xi_i),$$

and we say $\xi_i$'s are *Chern roots* of $\mathscr{E}$.

**Corollary 3.1.4** Let $\mathscr{E}, \mathscr{F}$ are locally free sheaf, with Chern roots $\xi_i, \eta_j$ respectively. Then

- $\mathscr{E} \otimes \mathscr{F}$ has Chern roots $\xi_i + \eta_j$.
- $\mathscr{H}om(\mathscr{F}, \mathscr{E})$ has Chern roots $\xi_i - \eta_j$.
- $\bigwedge^p \mathscr{E}$ has Chern roots $\sum_{k=1}^{p} \xi_{i_k}$, where $i_1 < i_2 < \cdots < i_p$.

## 3.2 Hirzebruch-Riemann-Roch

Suppose $\mathscr{E}$ is a locally free sheaf of rank $r$ on $X$, with Chern roots $\xi_i$.

**Definition 3.2.1 (Chern character)** The *Chern character* $\mathrm{ch}(\mathscr{E}) \in A^\bullet_{\mathbb{Q}}(X)$ is defined by

$$\mathrm{ch}(\mathscr{E}) = \sum_{i=1}^{r} e^{\xi_i},$$

where $e^\xi = \sum_{n\geq 0} \frac{\xi^n}{n!}$ for $\xi \in A^1(X)$.

**Definition 3.2.2 (Todd class)** The *Todd class* $\mathrm{td}(\mathscr{E}) \in A^\bullet_{\mathbb{Q}}(X)$ is defined by

$$\mathrm{td}(\mathscr{E}) = \prod_{i=1}^{r} \frac{\xi_i}{1 - e^{\xi_i}},$$

where

$$\frac{\xi}{1-e^\xi} = 1 + \frac{1}{2}\xi + \sum_{n\geq 1} (-1)^{n-1} \frac{B_n}{(2n)!}\xi^{2n},$$

and $B_n$ is the $n$-th Bernoulli number.

It is tedious to calculate general Chern character and Todd class. For example, if $\mathscr{E}$ has $i$-th Chern class $c_i$, then

$$\mathrm{ch}(\mathscr{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$
$$\mathrm{td}(\mathscr{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots$$

**Theorem 3.2.3 (Hirzebruch-Riemann-Roch)** For a locally free sheaf $\mathscr{E}$ over a smooth complete variety $X$ of dimension $n$, and suppose $\mathscr{T}$ is the tangent sheaf of $X$ (dual of the sheaf $\Omega_{X/\mathbb{C}}$ of differentials), we have

$$\chi(\mathscr{E}) = \int_X \mathrm{ch}(\mathscr{E}) \cdot \mathrm{td}(\mathscr{T}),$$

where the integral symbol means taking the degree of the component of degree $n$ in $A^\bullet_{\mathbb{Q}}(X)$.

*Remark.* There is a relative version of Riemann-Roch, which is stated by Grothendieck in 1957.

**Example 3.2.4 (algebraic curves)** Assume $X$ is a smooth projective curve, with canonical divisor $K$. Then $\mathscr{T} \simeq \mathscr{O}_X(-K)$, so

$$\operatorname{td}(\mathscr{T}) = 1 - \frac{1}{2}K.$$

For an invertible sheaf $\mathscr{O}_X(D)$, we have $\operatorname{ch}(\mathscr{O}_X(D)) = 1 + D$, so the HRR is just

$$\chi(\mathscr{O}_X(D)) = \int_X (1+D)\left(1 - \frac{1}{2}K\right) = \deg(D - K/2) = \deg D + 1 - g,$$

where $g$ is the genus of $X$, and this is the classical Riemann-Roch for curves.

# Chapter 4

# Chow Groups of Toric Varieties

We now compute the Chow group $A_\bullet(X_\Sigma)$ for some fan $\Sigma$.

## 4.1 Chow Groups of Tori

Recall that the Weil class group of an affine space is always trivial, since a polynomial ring over field is a UFD (see [1, pp132]).

**Proposition 4.1.1** $A_0(\mathbb{A}^n) = 0$ for $n > 0$.

*Proof.* For a closed point $p$ in $\mathbb{A}^n$, there is a line $L \simeq \mathbb{A}^1$ passing through $p$ when $n > 0$. Since the class group of $L$ is trivial, there is a rational function $f$ on $L$ such that $p = \mathrm{div}_L(f)$, which shows that $p$ is rationally equivalent to 0. Thus $A_0(\mathbb{A}^n) = 0$ for $n > 0$. □

However, for a general integral closed subvariety $Z$ of $\mathbb{A}^n$, it is hopeless to find a $Y \subset \mathbb{A}^n$ on which $Z$ is a principal divisor. We may use the big theorem 1.4.4 to deduce the isomorphism

$$A_k(\mathbb{A}^n) \simeq A_0(\mathbb{A}^{n-k}) = 0, \quad k < n,$$

since $\mathbb{A}^n$ is a trivial bundle of rank $k$ over $\mathbb{A}^{n-k}$. We would rather give another proof here.

**Lemma 4.1.2** For any variety $X$ the flat pullback

$$A_k(X) \to A_{k+1}(X \times \mathbb{A}^1),$$

is surjective.

*Proof.* Let $V \subset X \times \mathbb{A}^1$ be an integral closed subvariety of dimension $k+1$, and $W = \overline{p(V)}$ be the Zariski closure of the projection of $V$ in $X$. One can easily see that $\dim W = k$ or $k+1$:

- If $\dim W = k$, then $p^*W = W \times \mathbb{A}^1$ is a $(k+1)$-dimensional closed subvariety contains $V$. In this case we have $V = p^*W$.
- If $\dim W = k+1$, then by the theory of Weil divisors, the pullback

  $$A_k(W) \to A_{k+1}(W \times \mathbb{A}^1),$$

  is an isomorphism. In this case, $[V] \in A_{k+1}(W \times \mathbb{A}^1)$ corresponds to a subvariety $Z \subset W$ with $[V] = p^*[Z]$.

Above all, $p^*$ is surjective. □

By the above lemma, the composition

$$A_0(\mathbb{A}^{n-k}) \to A_1(\mathbb{A}^{n-k+1}) \to \cdots \to A_k(\mathbb{A}^n),$$

is surjective, and it follows that

**Theorem 4.1.3** $A_k(\mathbb{A}^n) = \begin{cases} 0, & k \neq n, \\ \mathbb{Z} \cdot [\mathbb{A}^n], & k = n. \end{cases}$

**Corollary 4.1.4** For an $n$-torus $T$, we have

$$A_k(T) = \begin{cases} 0, & k \neq n, \\ \mathbb{Z} \cdot [T], & k = n. \end{cases}$$

*Proof.* There is an affine space $\mathbb{A}^n$ such that $T$ is an open subvariety of $\mathbb{A}^n$. The flat pullback $A_k(\mathbb{A}^n) \to A_k(T)$ is surjective, yields that $A_k(T) = 0$ for $k < n$ by above theorem. □

## 4.2 Chow Groups of $X_\Sigma$

Now suppose $T$ is an $n$-torus with dual lattices $M, N$, and $\Sigma$ is a fan in $N_\mathbb{R}$. From [4, pp172], we know that there is an exact sequence for class group:

$$M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma) \to 0.$$

In fact, there is a similar exact sequence for general Chow groups. Note that there is a filtration

$$\varnothing = X_{-1} \subset X_0 \subset \cdots \subset X_n = X_\Sigma,$$

where $X_i = \bigcup_{\sigma \in \Sigma(n-i)} V(\sigma)$.

> **Proposition 4.2.1** Then the group $A^k(X_\Sigma) = A_{n-k}(X)$ is generated by $\{[V(\sigma)] : \sigma \in \Sigma(k)\}$.

*Proof.* Using localization sequence 1.4.5, we have the exact sequence

$$A_k(X_{i-1}) \to A_k(X_i) \to A_k(X_i - X_{i-1}) \to 0.$$

Note that

$$X_i - X_{i-1} = \bigcup_{\sigma \in \Sigma(n-i)} O(\sigma),$$

is a disjoint union of $i$-tori, for $k < i$ we have

$$A_k(X_i - X_{i-1}) = \bigoplus_{\sigma \in \Sigma(n-i)} A_k(O(\sigma)) = 0.$$

Let $i = k+1, k+2, \cdots, n$ we then have surjections

$$A_k(X_k) \to A_k(X_{k+1}) \to \cdots \to A_k(X_n) = A_k(X_\Sigma).$$

Note that $X_k$ has irreducible components $V(\sigma)$ for $\sigma \in \Sigma(n-k)$, so

$$A_k(X_k) = \bigoplus_{\sigma \in \Sigma(n-k)} \mathbb{Z} \cdot [V(\sigma)].$$

So $A_k(X_\Sigma)$ can be generated by $\{[V(\sigma)] : \sigma \in \Sigma(k)\}$. □

Thus, $A_\bullet(X_\Sigma)$ is generated by *$T$-invariant cycles*, i.e. cycles of form $\sum_{\sigma\in\Sigma} a_\sigma \cdot V(\sigma)$. And our next step is to determine when a $T$-invariant cycle is rationally equivalent to 0.

> **Proposition 4.2.2** Suppose $\sigma \in \Sigma(k)$. An element $m \in M(\sigma)$ can be identified with a nonzero rational function on $V(\sigma)$, and
>
> $$\mathrm{div}_{V(\sigma)}(m) = \sum_{\sigma \prec \tau \in \Sigma(k+1)} \langle m, u_{\tau,\sigma} \rangle \cdot V(\tau).$$
>
> Here $u_{\tau,\sigma} \in \sigma$ represents the ray generator of $\overline{\tau} \in \mathrm{Star}(\sigma)$.

*Proof.* By orbit-cone correspondence, we know that $V(\sigma)$ is a toric variety associated to fan

$$\mathrm{Star}(\sigma) = \{\overline{\tau} \subset N(\sigma)_{\mathbb{R}} : \sigma \prec \tau \in \Sigma\},$$

whose torus is $O(\sigma)$ with dual lattices

$$M(\sigma) = \sigma^{\perp} \cap M, \text{ and } N(\sigma) = N/\mathrm{span}(\sigma \cap N).$$

So it is just [4, pp171]. □

> **Theorem 4.2.3** There is an exact sequence
>
> $$\bigoplus_{\tau \in \Sigma(k-1)} M(\sigma) \to \mathbb{Z}^{\Sigma(k)} \to A^k(X_\Sigma) \to 0.$$
>
> The first map sends a rational function on $V(\sigma)$ to its principal divisor, and the second map is just taking the rationally equivalent class of a $T$-invariant cycle.

*Proof.* We only need to show that every $T$-invariant cycle $\alpha \in \mathbb{Z}^{\Sigma(k)}$ rationally equivalent to 0 comes from combinations of $\operatorname{div}_{V(\sigma)}(m)$. I can't prove this... $\square$

*Remark.* There is a general theorem about spherical varieties. See [3, Theorem 1].

**Proposition 4.2.4** Let $\Sigma_i$ be two fans in $(N_i)_{\mathbb{R}}$, $i = 1, 2$. The exterier product

$$A_\bullet(X_{\Sigma_1}) \otimes A_\bullet(X_{\Sigma_2}) \to A_\bullet(X_{\Sigma_1 \times \Sigma_2}),$$

is an isomorphism of graded groups.

*Proof.* Note that all cones of dimension $r$ in $\Sigma_1 \times \Sigma_2$ are of form $\sigma_1 \times \sigma_2$ with $\dim \sigma_1 + \dim \sigma_2 = r$, things are trivial by previous theorem. $\square$

**Example 4.2.5 (projective toric surface)** If $\Sigma$ is a smooth, complete fan in $\mathbb{R}^2$, then we may assume

$$\Sigma(1) = \{\rho_1, \cdots, \rho_r\},$$
$$\Sigma(2) = \{\sigma_1, \cdots, \sigma_r\},$$

where $\sigma_i = \rho_i + \rho_{i+1}$ $(\rho_{n+1} = \rho_1)$. Denoted by $D_i$ for $V(\rho_i)$, $\gamma_i$ for $V(\sigma_i)$, and $u_i = (a_i, b_i)$ for the ray generator of $\rho_i$. We know that $A_1(X_\Sigma)$ is generated by $D_i$, with relations

$$\sum_{i=1}^{r} a_i D_i = 0, \text{ and } \sum_{i=1}^{r} b_i D_i = 0.$$

For each $i$, there are just tow cones $\sigma_{i-1}, \sigma_i$ contains $\rho_i$. Since all cones are smooth, we can just choose

$$u_{\sigma_{i-1},\rho_i} = u_{i-1}, \text{ and } u_{\sigma_i,\rho_i} = u_{i+1}.$$

On the other hand, $\rho_i^\perp$ is generated by $m_i = (b_i, -a_i)$. Thus for each $i$ there is a relation in $A_0(X_\Sigma)$ determined by

$$\operatorname{div}_{V(\rho_i)}(m_i) = \langle m_i, u_{i-1}\rangle \gamma_{i-1} + \langle m_i, u_{i+1}\rangle \gamma_{i+1}.$$

Note that $\{u_{i-1}, u_i\}$ forms a basis of $\mathbb{Z}^2$, we then have $a_{i-1}b_i - a_ib_{i-1} = 1$ for $i = 1, \cdots, r$. Thus $A_0(X_\Sigma)$ is freely generated by each one of $\gamma_i$, and each pair $\gamma_i, \gamma_j$ are rationally equivalent.

In fact, the smooth condition is superfluous: the vectors in $\mathbb{R}^2$ defined by

$$v_{\sigma_{i-1},\rho_i} = \frac{1}{a_{i-1}b_i - a_ib_{i-1}} u_{i-1}, \text{ and } v_{\sigma_{i+1},\rho_i} = \frac{1}{a_ib_{i+1} - a_{i+1}b_i} u_{i+1},$$

will satisfy that $\langle m_i, v_{\sigma_{i-1},\rho_i}\rangle = \langle m_i, u_{\sigma_{i-1},\rho_i}\rangle$ and $\langle m_i, v_{\sigma_{i+1},\rho_i}\rangle = \langle m_i, u_{\sigma_{i+1},\rho_i}\rangle$, so

$$\operatorname{div}_{V(\rho_i)}(m_i) = \gamma_{i-1} - \gamma_{i+1}, \quad i = 1, \cdots, r.$$

# Chapter 5

# Toric Intersection Theory

## 5.1 Intersection with Cartier Divisors

For a $T$-invariant Cartier divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ on $X_\Sigma$, it associates to data $\{m_\sigma\}_{\sigma \in \Sigma}$, where $m_\sigma \in M$ such that

$$D|_{U_\sigma} + \mathrm{div}_{U_\sigma}(m_\sigma) = 0.$$

One can easily check that $m_\sigma$ is unique modulo $M(\sigma)$. What we want to do is to determine the intersection $D.\alpha$ for a $T$-invariant cycle.

**Proposition 5.1.1** Suppose $\tau \in \Sigma$, $V(\tau) \not\subset \mathrm{supp}(D)$ iff $a_\rho = 0$ for each $\rho \in \tau(1)$. In this case $m_\sigma \in M(\tau)$ for each $\sigma \in \Sigma$, and the restriction $D|_{V(\tau)}$ is well-defined, with data $\{m_\sigma\}_{\overline{\sigma} \in \mathrm{Star}(\tau)}$.

*Proof.* Straightforward. □

**Corollary 5.1.2** If $a_\rho = 0$ for each $\rho \in \tau(1)$, then

$$D.V(\tau) = -\sum \langle m_\sigma, u_{\sigma,\tau} \rangle [V(\sigma)],$$

where the sum runs over all cones $\sigma \in \Sigma$ such that $\tau$ is a facet of $\sigma$.

Now what if $a_\rho \neq 0$ for some $\rho \in \tau(1)$? We expect to find some $m \in M$ such that $D + \mathrm{div}_{X_\Sigma}(m)$ will satisfy the condition.

**Lemma 5.1.3** Let $D' = D + \mathrm{div}_{X_\Sigma}(m_\tau)$, then $V(\tau) \not\subset \mathrm{supp}(D')$.

*Proof.* Note that $D'$ has data $\{m_\sigma - m_\tau\}_{\sigma \in \Sigma}$, and one can directly show that $\langle m_\sigma - m_\tau, u_\rho \rangle = 0$ for $\rho \in \tau(1)$. □

Then we can deduce the general formula.

**Theorem 5.1.4** For $\tau \in \Sigma$ we have

$$D.V(\tau) = -\sum \langle m_\sigma - m_\tau, u_{\sigma,\tau} \rangle [V(\sigma)],$$

where the sum runs over all cones $\sigma \in \Sigma$ such that $\tau$ is a facet of $\sigma$.

## 5.2 Simplicial Case

Recall that a simplicial cone is a cone $\sigma$ with $\dim \sigma = |\sigma(1)|$, and a simplicial fan refers a fan consists of simplicial cones. Also note that in a simplicial toric variety, each Weil divisor is $\mathbb{Q}$-Cartier (c.f. [4, pp180]).

**Definition 5.2.1 (multiplicity of simplicial cones)** Let $\sigma$ is a simplicial cone in $N_\mathbb{R}$, with ray generators $u_1, \cdots, u_r$. Then $\bigoplus_{i=1}^r \mathbb{Z} u_i$ is a subgroup of $\mathrm{span}(\sigma) \cap N$ of finite index $\mathrm{mult}(\sigma)$, which is called the *multiplicity* of $\sigma$.

*Remark.* $\mathrm{mult}(\sigma) = 1$ iff $\sigma$ is smooth.

**Lemma 5.2.2** Let $\sigma$ is a simplicial cone in $N_\mathbb{R}$, with ray generators $u_1, \cdots, u_r$. For the facet $\tau = \mathrm{cone}(u_2, \cdots, u_r)$, one can pick $u_{\sigma,\tau} = \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)} u_1$.

*Proof.* We may assume $u_{\sigma,\tau} \in \sigma \cap N$ first. Combine $u_{\sigma,\tau}$ with a basis of $N_\tau = \text{span}(\tau) \cap N$ we can get a basis of $N_\sigma$. One can then see that there is a unique (positive) integer $a$ such that

$$u_1 = au_{\sigma,\tau} + v, \quad v \in N_\tau.$$

By considering the sublattices

$$\bigoplus_{i=1}^{r} \mathbb{Z}u_i \subset \mathbb{Z}u_1 + N_\tau \subset \mathbb{Z}u + N_\tau = N_\sigma,$$

one can see that $a$ is the index of $\mathbb{Z}u_1 + N_\tau$ in $\mathbb{Z}u + N_\tau$, which is $\frac{\text{mult}(\sigma)}{\text{mult}(\tau)}$. Thus $u_{\sigma,\tau} = \frac{\text{mult}(\sigma)}{\text{mult}(\tau)}u_1 + v$. Since $v \in N_\tau$ does not influence the projection in $N(\tau)$, we can then pick $u_{\sigma,\tau} = \frac{\text{mult}(\tau)}{\text{mult}(\sigma)}u_1$ as desired. □

**Theorem 5.2.3** Suppose $\Sigma$ is a simplicial fan. For $\tau \in \sigma$ and $\rho \in \Sigma(1) - \tau(1)$, we have

$$V(\rho).V(\tau) = \begin{cases} \frac{\text{mult}(\tau)}{\text{mult}(\sigma)}[V(\sigma)], & \sigma = \rho + \tau \in \Sigma, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Suppose $\rho$ has ray generator $u$ and $\tau$ has ray generators $u_1, \cdots, u_r$. Since $V(\rho)$ is $\mathbb{Q}$-Cartier, we can find a positive integer $l$ such that $l \cdot V(\rho)$ is Cartier, with data $\{m_\sigma\}_{\sigma\in\Sigma}$.

If $\sigma = \rho + \tau \in \Sigma$, then by definition we have $\langle m_\sigma, u\rangle = -l$. Thus $\langle m_\sigma, u_{\sigma,\tau}\rangle = -l$ by 5.2.2. For $\sigma' \neq \sigma$ such that $\tau$ is a facet of $\sigma'$, there is a ray $\rho'$ with generator $u'$ such that $\sigma' = \rho' + \tau$. By definition we also have $\langle m_{\sigma'}, u'\rangle = 0$, hence $\langle m_{\sigma'}, u_{\sigma',\tau}\rangle = 0$. Note that $V(\tau) \not\subset V(\rho)$, by 5.1.2 we can deduce that

$$(l \cdot V(\rho)).V(\tau) = l\frac{\text{mult}(\tau)}{\text{mult}(\sigma)}[V(\sigma)],$$

i.e. $V(\rho).V(\tau) = \frac{\text{mult}(\tau)}{\text{mult}(\sigma)}[V(\sigma)]$.

Otherwise $\langle m_\sigma, u'\rangle = 0$ for any $\sigma \succ \tau$ and ray $\rho'$ with generator $u'$ such that $\sigma = \rho' + \tau$, so $V(\rho).V(\tau) = 0$. □

*Remark.* If $\sigma = \rho + \tau \in \Sigma$, then $V(\rho) \cap V(\tau) = V(\sigma)$ is a proper intersection. If moreover $\sigma$ is smooth, then $V(\rho), V(\tau)$ intersect transversally. If $\rho + \tau \notin \Sigma$, then $V(\rho) \cap V(\tau) = \varnothing$.

**Example 5.2.4 (projective toric surface)** Recall example 4.2.5. Let $\Sigma$ be a complete fan in $\mathbb{R}^2$ (hence simplicial), and assume

$$\Sigma(1) = \{\rho_1, \cdots, \rho_r\},$$
$$\Sigma(2) = \{\sigma_1, \cdots, \sigma_r\},$$

as before. Note that $A_0(X_\Sigma) = \mathbb{Z} \cdot [\gamma]$ where $[\gamma] \sim [V(\sigma_i)]$ for each $i$. By above theorem, for $i \neq j$ we have

$$D_i.D_j = \begin{cases} \frac{1}{|a_ib_j - a_jb_i|}[\gamma], & |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

So it remains to compute the "self intersection" $D_i.D_i$. Note that $\langle u_i, u_i \rangle \neq 0$, then we have

$$D_i = \frac{1}{\langle u_i, u_i \rangle} \left( \operatorname{div}(m) - \sum_{j \neq i} \langle u_i, u_j \rangle D_j \right).$$

Thus

$$\begin{aligned} D_i.D_i &= D_i.\frac{1}{\langle u_i, u_i \rangle} \left( \operatorname{div}(m) - \sum_{j \neq i} \langle u_i, u_j \rangle D_j \right) \\ &= -\frac{1}{a_i^2 + b_i^2} \sum_{j \neq i} (a_ia_j + b_ib_j) D_i.D_j \\ &= -\frac{1}{a_i^2 + b_i^2}((a_ia_{i-1} + b_ib_{i-1})D_i.D_{i-1} + (a_ia_{i+1} + b_ib_{i+1})D_i.D_{i+1}) \\ &= -\frac{1}{(a_i^2 + b_i^2)} \left( \frac{a_ia_{i-1} + b_ib_{i-1}}{|a_ib_{i-1} - a_{i-1}b_i|} + \frac{a_ia_{i+1} + b_ib_{i+1}}{|a_ib_{i+1} - a_{i+1}b_i|} \right) [\gamma]. \end{aligned}$$

From this, one can check that $X_\Sigma$ has rational Chow ring

$$A^\bullet_{\mathbb{Q}}(X_\Sigma) = \frac{\mathbb{Q}[x_1, \cdots, x_i]}{\left(\sum_{i=1}^r a_ix_i, \sum_{i=1}^r b_ix_i, \{x_ix_j\}_{|i-j|>1}\right)}.$$

**Example 5.2.5 (quadratic cone)** In real plane $\mathbb{R}^2$, the triangle $T$ with vertices $(0,0),(2,0),(0,1)$ is a very ample (in fact normal) polytope. The associated toric variety is just the weighted projective plane $Q = \mathbb{P}(1,1,2)$, and $T$ gives a closed immersion

$$\begin{aligned} Q &\to \mathbb{P}^3 \\ [a,b,c] &\mapsto [a^2, b^2, ab, c] \end{aligned}$$

realizing $Q$ as the quadratic cone $xy = z^2$ in $\mathbb{P}^3$.

In the normal fan $\Sigma_T$ of $T$, there are 3 ray generators $u_1 = (1,0), u_2 = (0,1), u_3 = (-1,-2)$. The 3 rays correspond to 3 divisors $D_1, D_2, D_3$. Applying previous discussion to the normal fan $\Sigma_T$, one can see that $A_1(Q)$ is freely generated by $D_2$, with self intersection $D_2.D_2 = 2[\gamma]$.

In $\mathbb{P}^3$, $D_1$ is the line $y = z = 0$, $D_2$ is the conic (which is a hyperplane section) $w = 0, xy = z^2$, and $D_3$ is the line $x = z = 0$. One can see that $D_1, D_2$ intersect transversally at $[1,0,0,0]$, and $D_1.D_2 = [\gamma]$ as expected. One can also see that $D_1, D_3$ intersect properly at the singularity $[0,0,0,1]$, but $D_1.D_3 = \frac{1}{2}[\gamma]$ is not an integral cycle. Note that $D_1 \sim D_3$ are not Cartier, while $2D_1 \sim 2D_3 \sim D_2$ are.

This is an example from [1, pp428], which shows that the intersection on a singular variety may not behave well.

## 5.3 Chow Rings of Smooth Complete Toric Varieties

Now suppose $\Sigma$ is a smooth complete fan, we now come to our goal to compute $A^\bullet(X_\Sigma)$. By the discussion in simplicial case, we have

**Lemma 5.3.1** For distinct rays $\rho_1, \cdots, \rho_r$ in $\Sigma(1)$, we have

$$[V(\rho_1)] \cdot [V(\rho_2)] \cdots [V(\rho_2)] = \begin{cases} [V(\sigma)], & \sum_{i=1}^r \rho_i = \sigma \in \Sigma, \\ 0, & \text{otherwise}, \end{cases}$$

in the Chow ring $A^\bullet(X_\Sigma)$.

Also note that $A^\bullet(X_\Sigma)$ is generated by $\{V(\rho) : \rho \in \Sigma(1)\}$ as a commutative ring, so $A^\bullet(X_\Sigma)$ is a quotient of the *total coordinate ring* $\mathbb{Z}[x_\rho : \rho \in \Sigma(1)]$. We define

$$\mathcal{I} = \left( x_{\rho_1} \cdots x_{\rho_r} : \rho_1, \cdots, \rho_r \text{ are distinct rays in } \Sigma, \sum \rho_i \notin \Sigma \right),$$

to be the *Stanley-Reisner ideal.* And the principal divisors also generate the graded ideal

$$\mathcal{J} = \left( \sum_\rho \langle m, u_\rho \rangle x_\rho : m \in M \right).$$

**Definition 5.3.2** We say the quotient ring

$$R(\Sigma) := \frac{\mathbb{Z}[x_\rho : \rho \in \Sigma(1)]}{\mathcal{I} + \mathcal{J}},$$

is the *Chow ring* of fan $\Sigma$.

One can see that there is a surjective morphism $R(\Sigma) \to A^\bullet(X_\Sigma)$ defined by $x_\rho \mapsto [V_\rho]$. For $\tau \in \Sigma(k), m \in M(\tau)$ we have

$$\begin{aligned} & \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle x_\rho \prod_{\varrho \in \tau(1)} x_\varrho \\ = & \sum_{\rho \in \Sigma(1) - \tau(1)} \langle m, u_\rho \rangle x_\rho \prod_{\rho \neq \varrho \in \tau(1)} x_\varrho \\ = & \sum_{\rho + \tau \in \Sigma(k+1)} \langle m, u_\rho \rangle \prod_{\varrho \in (\rho + \tau)(1)} x_\varrho \end{aligned}$$

in $R(\Sigma)$. It follows that

**Theorem 5.3.3** $R(\Sigma) \simeq A^\bullet(X_\Sigma)$ as graded rings.

*Remark.* We also have $A^\bullet(X_\Sigma) \simeq H^{2\bullet}(X_\Sigma^{\text{an}}, \mathbb{Z})$, where $X_\Sigma^{\text{an}}$ carries the analytic topology. And one can check that

$$\operatorname{rank}(A_k(X_\Sigma)) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} N_i,$$

where $n = \text{rank}(N)$ and $N_i = |\Sigma(i)|$. It is surprising that such a rank only depends on $N_i$.

**Example 5.3.4** $A^\bullet(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1})$.

**Example 5.3.5** $A^\bullet(\mathbb{F}_r) = \mathbb{Z}[h,\zeta]/(h^2, \zeta^2 + rh\zeta)$.

*Remark.* It can be shown that if $\Sigma$ is simplicial and complete, then

$$A^\bullet_{\mathbb{Q}}(X_\Sigma) = R(\Sigma) \otimes \mathbb{Q}.$$

See [4, pp616-617].

# Chapter 6

# Toric Riemann-Roch

To be continued...

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