7. Sheaf cohomology of toric varieties

7.1. **Some backgrounds.** Roughly speaking, the cohomology of a sheaf $\mathscr{F}$ on a variety X is the right derived functor of $\Gamma(X, \mathscr{F})$.

7.1.1. *Computation.* Although the abstract definition of the sheaf cohomology has nice properties, it's not useful for explicit computation. One down-earth way of viewing sheaf cohomology is to use Čech complex. To be precise, given an open covering $\mathfrak{U} = \{U_i\}$ of X, the p-th Čech cochain is

$$\check{C}^p(\mathfrak{U}, \mathscr{F}) = \bigoplus_{(i_0,\dots,i_p)\in[\ell]_p} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

For $\alpha \in \check{C}^p(\mathfrak{U}, \mathscr{F})$, the differential is given by

$$\mathrm{d}^p(\alpha)(i_0, \dots, i_{p+1}) = \sum_{k=0}^{p+1} (-1)^k \alpha(i_0, \dots, \widehat{i}_k, \dots, i_{p+1})|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

The Čech cochain together with above differential is a complex, and the p-th Čech cohomology $\check{H}(\mathfrak{U}, \mathscr{F})$ is defined to be the cohomology of this complex.

Theorem 7.1.1 (Serre)**.** Let $\mathscr{F}$ be a quasi-coherent sheaf on an affine variety U. Then $H^p(U, \mathscr{F}) = 0$ for all $p > 0$.

By a standard argument of spectral sequence, this suggests the following result.

Theorem 7.1.2. Let $\mathfrak{U}$ be an affine open covering of a variety X and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there are natural isomorphisms

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \cong H^p(X, \mathscr{F})$$

for all $p \geq 0$.

7.1.2. *Higher direct image.* Given a morphism $f : X \to Y$ of varieties and a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on X, the direct image is the sheaf $f_*\mathscr{F}$ on Y defined by

$$U \mapsto \mathscr{F}(f^{-1}(U)).$$

It's clear there is an isomorphism $H^0(Y, f_*\mathscr{F}) = H^0(X, \mathscr{F})$ since $f^{-1}(Y) = X$. More generally, there are natural homomorphisms $H^p(Y, f_*\mathscr{F}) \to H^p(X, \mathscr{F})$, which may not be an isomorphism. One obstruction comes from the higer direct image $R^p f_*\mathscr{F}$, which is the sheaf on Y associated to the presheaf defined by

$$U \mapsto H^p(f^{-1}(U), \mathscr{F}).$$

Proposition 7.1.1. Suppose $f : X \to Y$ is a morphism and $\mathscr{F}$ is a quasi-coherent sheaf on X such that $R^q f_*\mathscr{F} = 0$ for all $q > 0$. Then

$$H^p(Y, f_*\mathscr{F}) \cong H^p(X, \mathscr{F})$$

for all $p \geq 0$.

7.1.3. *Serre's result.*

Theorem 7.1.3 (Serre vanishing)**.** Let $\mathscr{L}$ be an ample line bundle on a projective variety X. Then for any coherent sheaf $\mathscr{F}$ on X, one has

$$H^p(X, \mathscr{F} \otimes \mathscr{L}^{\otimes \ell}) = 0$$

for all $p > 0$ and $\ell \gg 0$.

Theorem 7.1.4 (Serre duality)**.** Let ω_X be the canonical sheaf of a complete normal Cohen-Macaulay variety X of dimension n. Then for every locally free sheaf $\mathscr{F}$ of finite rank on X, there are natural isomorphisms

$$H^p(X, \mathscr{F})^\vee \cong H^{n-p}(X, \omega_X \otimes_{\mathcal{O}_X} \mathscr{F}^\vee).$$

7.2. **Cohomology of toric divisors.** For Čech cohomology of toric variety, there is an obvious choice of affine open covering, that is, $\mathfrak{U} = \{U_\sigma\}_{\sigma \in \Sigma_{\max}}$, where $\Sigma_{\max}$ is the set of maximal cones in Σ. Given a torus-invariant Cartier divisor $D = \sum_\rho a_\rho D_\rho$, the Čech complex is given by

$$\check{C}^p(\mathfrak{U}, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\gamma = (i_0, \ldots, i_p) \in [\ell]_p} H^0(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}_{X_\Sigma}(D)).$$

For convenience we always write $\sigma_\gamma = \sigma_{i_0} \cap \cdots \cap \sigma_{i_1}$, and denote

$$\check{C}^p(\mathfrak{U}, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\gamma \in [\ell]_p} H^0(U_{\sigma_\gamma}, \mathcal{O}_{X_\Sigma}(D)).$$

By Proposition 4.4.1, there is a grading on the cohomology as follows

$$H^0(U_\sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_m H^0(U_\sigma, \mathcal{O}_{X_\Sigma}(D))_m,$$

where for $m \in M$,

$$H^0(U_\sigma, \mathcal{O}_{X_\Sigma}(D))_m = \begin{cases} \mathbb{C} \cdot \chi^m, & \langle m, u_\rho \rangle \geq -a_\rho \\ 0, & \text{otherwise.} \end{cases}$$

Thus it suffices to compute the Čech cohomology for each weight $m \in M$.

Theorem 7.2.1. Let $D = \sum_\rho a_\rho D_\rho$ be a Weil divisor on X_Σ. Fix $m \in M$ and $p \geq 0$.

(1) $H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m \cong \widetilde{H}^{p-1}(V_{D,m}, \mathbb{C})$, where $V_{D,m} = \bigcup_{\sigma \in \Sigma} \operatorname{Conv}\{u_\rho \mid \rho \in \sigma(1), \langle m, u_\rho \rangle \geq -1\}$.
(2) If D is $\mathbb{Q}$-Cartier, then $H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))_m \cong \widetilde{H}^{p-1}(V_{D,m}^{\text{supp}}, \mathbb{C})$, where $V_{D,m}^{\text{supp}} = \{u \in |\Sigma| \mid \langle m, u \rangle < \varphi_D(u)\}$ and φ_D is support function of D.

7.3. **Vanishing theorems I.**

7.4. **Vanishing theorems II.**