6. Canonical divisors of toric variety

6.1. **Some backgrounds.** Suppose R is a $\mathbb{C}$-algebra. The module of Kähler differentials of R over $\mathbb{C}$, denoted by $\Omega^1_{R/\mathbb{C}}$, is the R-module generated by the formal symbols $\mathrm{d}f$ for $f \in R$, modulo the relations

(1) $\mathrm{d}(cf+g) = c\mathrm{d}f + \mathrm{d}g$ for all $c \in \mathbb{C}, f, g \in R$.
(2) $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ for all $f, g \in R$.

Example 6.1.1. If $R = \mathbb{C}[x_1, \dots, x_n]$, then

$$\Omega^1_{R/\mathbb{C}} \cong \bigoplus_{i=1}^{n} R\mathrm{d}x_i.$$

Proposition 6.1.1. Let R_f be the localization of $\mathbb{C}$-algebra R with respect to the non-nilpotent element $f \in R$. Then $\Omega^1_{R_f/\mathbb{C}} \cong \Omega^1_{R/\mathbb{C}} \otimes R_f$.

Let X be a variety. Then the cotangent sheaf Ω^1_X is the sheaf of $\mathcal{O}_X$-modules defined by

$$\Omega^1_X(U) = \Omega_{\mathcal{O}^1_X(U)/\mathbb{C}}$$

on affine open subsets U, and the tangent sheaf $\mathscr{T}_X$ is the dual sheaf of Ω^1_X. More generally, the sheaf of p-forms is defined by $\bigwedge^p \Omega^1_X$.

A non-trivial fact is that X is smooth if and only if Ω^1_X is locally free, and thus Ω^p_X is locally free when X is smooth. But the sheaf of p-forms may behave badly when X is just normal, thought it's locally free on the smooth locus $j\colon U \hookrightarrow X$. One way to handle this is to consider the Zariski p-forms, defined by $\widehat{\Omega}^p_X := j_*\Omega^p_X|_U$, which gives a reflexive version of Ω^p_X.

For a reflexive sheaf $\mathscr{L}$ of rank one, there exists some Weil divisor D such that $\mathscr{L} \cong \mathcal{O}_X(D)$. In particular, for a normal variety X, the canonical sheaf

$$\omega_X = \widehat{\Omega}^n_X$$

is a reflexive sheaf of rank one, and the Weil divisor corresponding to ω_X is called the canonical divisor, denoted by K_X.

6.2. **One-forms on toric varieties.** In this section we wil study the sheaves $\Omega^1_{X_\Sigma}$ and $\widehat{\Omega}^1_{X_\Sigma}$ on a normal toric variety.

6.2.1. *The first exact sequence.* Firstly note that the coordinate ring of the torus T_N is the semigroup $\mathbb{C}[M]$. Then the map

$$\begin{aligned} \Omega_{\mathbb{C}[M]/\mathbb{C}} &\to M \otimes_{\mathbb{Z}} \mathbb{C}[M] \\ \mathrm{d}\chi^m &\mapsto m \otimes \chi^m \end{aligned}$$

gives an isomorphism of $\mathbb{C}[M]$-modules, and thus

$$\Omega^1_{T_N} \cong M \otimes_{\mathbb{Z}} \mathcal{O}_{T_N}.$$

Remark 6.2.1. As a consequence, $\mathrm{d}\chi^m/\chi^m$ is a global section of $\Omega^1_{T_N}$ that maps to $m \otimes 1$, and hence is invariant under the action of T_N.

Now consider the toric variety X_Σ. For $\rho \in \Sigma(1)$, the inclusion $i\colon D_\rho \hookrightarrow X_\Sigma$ gives the sheaf $i_*\mathcal{O}_{D_\rho}$ on X, and for convenience we just denote it by $\mathcal{O}_{D_\rho}$. Using the map $M \to \mathbb{Z}$ given by $m \mapsto \langle m, u_\rho\rangle$, there is the following composition

$$M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \to \mathcal{O}_{X_\Sigma} \to \mathcal{O}_{D_\rho}.$$

This gives a natural map

$$\beta\colon M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{D_\rho}.$$

On the other hand, on the affine piece $U_\sigma = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$, one can define

$$\begin{aligned} \Omega^1_{\mathbb{C}[\sigma^\vee\cap M]/\mathbb{C}} &\to M \otimes_{\mathbb{Z}} \mathbb{C}[\sigma^\vee \cap M] \\ \mathrm{d}\chi^m &\mapsto m \otimes \chi^m, \end{aligned}$$

and these affine pieces patch together to give a map $\alpha\colon \Omega^1_{X_\Sigma} \to M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma}$.

Theorem 6.2.1. For a smooth toric variety X_Σ, the sequence

$$0 \to \Omega^1_{X_\Sigma} \xrightarrow{\alpha} M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \xrightarrow{\beta} \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{D_\rho} \to 0.$$

is exact.

Proof. Firstly let's show $\beta \circ \alpha = 0$. On the affine piece $U_\sigma \subseteq X_\Sigma$, one has $\mathcal{O}_{X_\Sigma}(-D_\rho)|_{U_\rho}$ is the ideal sheaf of the subvariety $D_\rho \cap U_\sigma$ by Proposition 4.1.1, and thus the subvariety $D_\rho \cap U_\rho$ is defined by the ideal

$$I_\rho = \Gamma(U_\sigma, \mathcal{O}_{X_\Sigma}(-D_\rho)) = \bigoplus_{\operatorname{div}(\chi^m)|_{U_\rho} \geq D_\rho|_{U_\rho}} \mathbb{C}\cdot\chi^m = \bigoplus_{\substack{m\in\sigma^\vee\cap M \\ \langle m,u_\rho\rangle\geq 1}} \mathbb{C}\cdot\chi^m.$$

Over U_σ, the composition $\Omega^1_{X_\Sigma} \to M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \to \mathcal{O}_{D_\rho}$ is given by sending 1-form $\mathrm{d}\chi^m, m \in \sigma^\vee \cap M$ to the equivalent class of $\langle m, u_\rho\rangle\chi^m$ in $\mathbb{C}[\sigma^\vee \cap M]/I_\rho$. It's clear zero if $\langle m, u_\rho\rangle = 0$, and if $\langle m, u_\rho\rangle \neq 0$, $\langle m, u_\rho\rangle\chi^m$ lies in I_ρ.

Now let's show the sequence is exact over affine piece U_σ. Since σ is smooth, we may assume $\sigma = \operatorname{Cone}(e_1, \dots, e_r)$, where $r \leq n$ and $e_1, \dots, e_n$ is a basis of N. Then $U_\sigma = \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$. Let $x_1, \dots, x_n$ denote the characters of the corresponding dual basis of M. The coordinate ring of U_σ is $R = \mathbb{C}[x_1, \dots, x_r, x^\pm_{r+1}, \dots, x^\pm_n]$. By Example 6.1.1 and Proposition 6.1.1, the 1-forms on U_σ is given by

$$\Omega^1_{R/\mathbb{C}} = \bigoplus_{i=1}^n R\mathrm{d}x_i.$$

Then the map α is given by

$$\begin{aligned} \alpha\colon \Omega^1_{R/\mathbb{C}} = \bigoplus_{i=1}^n R\mathrm{d}x_i &\to M \otimes_{\mathbb{Z}} R = \bigoplus_{i=1}^n R \\ \sum_{i=1}^n f_i\mathrm{d}x_i &\mapsto (f_1x_1, \dots, f_nx_n). \end{aligned}$$

This gives the exact sequence

$$0 \to \Omega^1_{R/\mathbb{C}} \to \bigoplus_{i=1}^{n} R \to \bigoplus_{i=1}^{r} R/\langle x_i \rangle \to 0$$

since $x_{r+1}, \ldots, x_n$ are units in R. This completes the proof. □

Remark 6.2.2. Replacing $\Omega^1_{X_\Sigma}$ by the Zariski 1-form $\widehat{\Omega}^1_X$, the same result still holds for simplical toric variety X_Σ.

6.2.2. *The Euler sequence.* For the projective space $\mathbb{P}^n$, there is a famous exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

called the Euler sequence. Here is a toric generalization of this result.

Theorem 6.2.2. Let X_Σ be a simplical toric variety with no torus factors. Then there is an exact sequence

$$0 \to \widehat{\Omega}^1_{X_\Sigma} \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(-D_\rho) \to \mathrm{Cl}(X_\Sigma) \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \to 0.$$

Moreover, if X_Σ is smooth, this reduces to

$$0 \to \Omega^1_{X_\Sigma} \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(-D_\rho) \to \mathrm{Pic}(X_\Sigma) \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \to 0.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{\Omega}^1_{X_\Sigma} & \longrightarrow & M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{D_\rho} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(-D_\rho) & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma} & \longrightarrow & \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{D_\rho} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Cl}(X_\Sigma) \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} & \longrightarrow & \mathrm{Cl}(X_\Sigma) \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It's clear the third row is exact, and the first row is exact by Remark 6.2.2. By Proposition 4.1.1 we have the following exact sequence

$$0 \to \mathcal{O}_{X_\Sigma}(-D_\rho) \to \mathcal{O}_{X_\Sigma} \to \mathcal{O}_{D_\rho} \to 0,$$

and thus the third row is exact. Since X_Σ has no torus factors, by Theorem 4.2.1 one has the middle column is exact. Then the five lemma yields the desired result. □

6.3. **Differential forms on toric varieties.**

6.3.1. *Properties of wedge product.*

Proposition 6.3.1. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of locally free sheaves on a variety X with $\operatorname{rk}\mathscr{F} = m$ and $\operatorname{rk}\mathscr{H} = n$. Then $\operatorname{rk}\mathscr{G} = m+n$ and there is an isomorphism

$$\bigwedge^{m+n}\mathscr{G} \cong \bigwedge^{m}\mathscr{F} \otimes_{\mathcal{O}_X} \bigwedge^{n}\mathscr{H}.$$

Corollary 6.3.1 (adjunction formula)**.** Let $Y \subseteq X$ be a smooth subvariety of a smooth variety X with $\dim Y = m$ and $\dim X = n$. Then

$$\omega_Y \cong \omega_X \otimes \bigwedge^{n-m} \mathscr{N}_{Y/X}.$$

Proof. Consider the following exact sequence

$$0 \to \mathscr{I}_Y/\mathscr{I}_Y^2 \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega_Y^1 \to 0.$$

□

6.3.2. *The canonical sheaf of toric varieties.*

Theorem 6.3.1. For a toric variety X_Σ, the canonical sheaf ω_{X_Σ} is given by

$$\omega_{X_\Sigma} \cong \mathcal{O}_{X_\Sigma}(-\sum_\rho D_\rho).$$

Thus $K_{X_\Sigma} = -\sum_\rho D_\rho$ is a torus-invariant canonical divisor on X_Σ.

Proof. For convenience here we only give a proof with the assumption X_Σ is smooth and without torus factors. By Theorem 6.2.2 there is the following exact sequence

$$0 \to \Omega^1_{X_\Sigma} \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(-D_\rho) \to \operatorname{Pic}(X_\Sigma)\otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \to 0.$$

Note that each $\mathcal{O}_{X_\Sigma}(-D_\rho)$ is a line bundle since X_Σ is smooth, and if we set $r = |\Sigma(1)|$, then it's easy to see $\operatorname{Pic}(X_\Sigma)\otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} = \mathcal{O}_{X_\Sigma}^{r-n}$. Thus by Proposition 6.3.1 one has

$$\bigwedge^{n}\Omega^1_{X_\Sigma} \otimes_{\mathcal{O}_{X_\Sigma}} \bigwedge^{r-n}\mathcal{O}^{r-n}_{X_\Sigma} \cong \bigwedge^{r}(\bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(-D_\rho)).$$

The right hand is isomorphic to

$$\bigotimes_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(-D_\rho) \cong \mathcal{O}_{X_\Sigma}(-\sum_{\rho\in\Sigma(1)} D_\rho)$$

The left hand is isomorphic to $\bigwedge^n \Omega^1_{X_\Sigma} = \omega_X$, since $\bigwedge^{r-n}\Omega^{r-n}_{X_\Sigma} \cong \mathcal{O}_{X_\Sigma}$. This completes the proof for the case X_Σ is smooth without torus factor. □

Example 6.3.1. The canonical bundle of $\mathbb{P}^n$ is

$$\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$$

for all $n \geq 1$ since $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ and thus $D_0 \sim D_1 \sim \cdots \sim D_n$.

Example 6.3.2. In Example 4.2.3, when we computed the class group of Hirzebruch surface, we wrote divisors D_ρ as $D_1, \ldots, D_4$ and showed that

$$D_3 \sim D_1$$
$$D_4 \sim rD_1 + D_2.$$

Thus the canonical bundle can be written as

$$\omega_{\mathscr{H}_r} = \mathcal{O}_{\mathscr{H}_r}(-D_1 - D_2 - D_3 - D_4) = \mathcal{O}_{\mathscr{H}_r}(-(r+2)D_1 - 2D_2).$$