4. Divisors on toric variety

4.1. **Some backgrounds.** Let X be an irreducible variety. A prime divisor $D \subseteq X$ is an irreducible subvariety of codimension one. Any prime ideal D will gives a subring of rational function field $\mathbb{C}(X)$ as

$$\mathcal{O}_{X,D} = \{\phi \in \mathbb{C}(X) \mid \phi \text{ is defined on } U \subseteq X \text{ open with } U \cap D \neq \varnothing\}.$$

A simple observation is that if $U \subseteq X$ is an open subset such that $U \cap D \neq \varnothing$, then $\mathcal{O}_{X,D} = \mathcal{O}_{U,U\cap D}$. This shows we can reduce to the affine case when we consider the ring $\mathcal{O}_{X,D}$. Suppose $X = \operatorname{Spec} R$ for an integral domain R and given a prime divisor $D = V(\mathfrak{p})$. Then

$$\mathcal{O}_{X,D} = R_{\mathfrak{p}}$$

is a local ring.

Example 4.1.1. Let X be the affine space $\mathbb{C} = \operatorname{Spec} \mathbb{C}[x]$ with rational function field $\mathbb{C}(x)$. Then the prime divisor $\{0\} = V(x)$ has the local ring

$$\mathcal{O}_{\mathbb{C},\{0\}} = \mathbb{C}[x]_{\langle x\rangle},$$

which is a DVR.

Remark 4.1.1. More generally, for a normal variety X, the local ring $\mathcal{O}_{X,D}$ for a prime divisor is a DVR.

For convenience, in the remaining of this section, we always assume the variety X is normal. Let $\operatorname{Div}(X)$ denote the free abelian group generated by the prime divisors on X. A Weil divisor is an element in $\operatorname{Div}(X)$. Moreover, any $f \in \mathbb{C}(X)^*$ gives a Weil divisor

$$\operatorname{div}(f) := \sum_D \nu_D(f) D,$$

where ν_D is the valuation of DVR $\mathcal{O}_{X,D}$ and the sum[4] is over all prime divisors $D \subseteq X$. Such a divisor is called a principal divisor, and the set of all principal divisors is denoted by $\operatorname{Div}_0(X)$.

Let $D = \sum_i a_i D_i$ is a Weil divisor on X and $U \subseteq X$ be an open subset. Then

$$D|_U = \sum_{U\cap D_i \neq \varnothing} a_i U \cap D_i$$

is a Weil divisor on U called the restriction of D on U.

A Weil divisor D on X is called a Cartier divisor if there exists an open covering $\{U_i\}_{i\in I}$ of X such that $D|_{U_i}$ is principal in U_i for each $i \in I$. If $D|_{U_i} = \operatorname{div}(f_i)$ for $i \in I$, then $\{(U_i, f_i)\}$ is called the local data for D. The group of Cartier divisors is denoted by $\operatorname{CDiv}(X)$.

The class group $\operatorname{Cl}(X)$ is defined by $\operatorname{Div}(X)/\operatorname{Div}_0(X)$ and the picard group is defined by $\operatorname{Pic}(X) = \operatorname{CDiv}(X)/\operatorname{Div}_0(X)$. In general these groups are not easy to compute, here we list some results.

[4]A fact is that the sum is finite.

Theorem 4.1.1. Let R be a UFD and $X = \operatorname{Spec} R$. Then $\operatorname{Cl}(X) = 0$.

Example 4.1.2. $\operatorname{Cl}(\mathbb{C}^n) = 0$ since $\mathbb{C}[x_1, \ldots, x_n]$ is a UFD.

Theorem 4.1.2. Let U be an open subset of a normal variety X and $D_1, \ldots, D_s$ be the irreducible components of $X \setminus U$ that are prime divisors. Then the following sequence is exact

$$\bigoplus_{j=1}^{s} \mathbb{Z}\, D_j \to \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0,$$

where the first map sends $\sum_j a_j D_j$ to its divisor class in $\operatorname{Cl}(X)$ and the second map is restriction.

For relations between Weil divisors and Cartier divisors, a fact is that if X is a smooth variety, then every Weil divisor is a Cartier divisor, and the converse statement also holds for a toric variety X_Σ, that if, if every Weil divisor of X_Σ is a Cartier divisor, then X_Σ is smooth.

Finally we introduce the sheaf of a Weil divisor D on X, denoted by $\mathcal{O}_X(D)$, which is defined by

$$U \mapsto \mathcal{O}_X(D) := \{f \in \mathbb{C}(X)^* \mid (\operatorname{div}(f)_D)|_U \geq 0\} \cup \{0\}.$$

In fact, it's a coherent $\mathcal{O}_X$-modules.

4.2. **Weil divisors on toric varieties.** Let X_Σ be the toric variety of fan in $N_\mathbb{R}$ with $\dim N_\mathbb{R} = n$. In this section we will use torus-invariant prime divisors and characters to give a lovely description of class group of X_Σ.

4.2.1. *The divisor of a character.* By the orbit-cone correspondence, $\rho \in \Sigma(1)$ gives the codimension one orbit $O(\rho)$ whose closure $\overline{O(\rho)}$ admits a codimension one toric subvariety structure by Proposition 3.2.1. Thus $\overline{O(\rho)}$ gives a T_N-invariant prime divisor on X_Σ. To emphasize that $\overline{O(\rho)}$ is a divisor we will denote it by D_ρ for convenience. Then D_ρ gives the DVR $\mathcal{O}_{X_\Sigma, D_\rho}$ with valuation

$$\nu_\rho \colon \mathbb{C}(X_\Sigma)^* \to \mathbb{Z}\,.$$

Recall that any ray $\rho \in \Sigma(1)$ has a minimal generator $u_\rho \in \rho \cap N$, and also note that when $m \in M$, the character $\chi^m \colon T_N \to \mathbb{C}^*$ is a rational function in $\mathbb{C}(X_\Sigma)^*$ since T_N is Zariski open in X_Σ.

Proposition 4.2.1. Let u_ρ be the minimal generator of ray $\rho \in \Sigma(1)$ and χ^m be a character corresponding to $m \in M$. Then

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle.$$

Proof. Firstly we extend u_ρ to a basis $e_1 = \rho, e_2, \ldots, e_n$ of N, and then we may assume $N = \mathbb{Z}^n$ and $\rho = \operatorname{Cone}(e_1) \subseteq \mathbb{R}^n$. Then the corresponding affine toric variety is

$$U_\rho = \operatorname{Spec}(\mathbb{C}[x_1, x_2^{\pm}, \ldots, x_n^{\pm}]) = \mathbb{C} \times (\mathbb{C}^*)^{n-1},$$

and $D_\rho \cap U_\rho$ is defined by $x_1 = 0$. Then as we have seen in Example 4.1.1, one has

$$\mathcal{O}_{X_\Sigma, D_\rho} = \mathcal{O}_{U_\rho, U_\rho \cap D_\rho} = \mathbb{C}[x_1, \dots, x_n]_{\langle x_1 \rangle}.$$

For $f \in \mathbb{C}(x_1, \dots, x_n)^*$, the valuation $\nu_\rho(f)$ is given by $\nu_\rho(f) = \ell$ when

$$f = x_1^\ell \frac{g}{h}, \quad g, h \in \mathbb{C}[x_1, \dots, x_n] \setminus \langle x_1 \rangle.$$

Then the formula for $\nu_\rho(\chi^m)$ can be seen from

$$\chi^m = x_1^{\langle m, e_1 \rangle} \cdots x_n^{\langle m, e_n \rangle} = x_1^{\langle m, u_\rho \rangle} \cdots x_n^{\langle m, e_n \rangle}.$$

□

Proposition 4.2.2. For $m \in M$, the divisor of character χ^m is given by

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

Proof. Note that D_ρ are irreducible components of $X \setminus T_N$, and χ^m is nonzero on T_N. Thus $\operatorname{div}(\chi^m)$ is supported on $\bigcup_{\rho \in \Sigma(1)} D_\rho$ and thus

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \nu_\rho(\chi^m) D_\rho = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

□

4.2.2. *Computing the class group.* Divisors of the form $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ are precisely the divisors invariant under the torus action. Thus

$$\operatorname{Div}_{T_N}(X_\Sigma) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \subseteq \operatorname{Div}(X)$$

is the group of T_N-invariant Weil divisors on X_Σ.

Theorem 4.2.1. There is the following exact sequence

$$M \to \operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to 0,$$

where the first map is $m \mapsto \operatorname{div}(\chi^m)$ and the second sends a T_N-invariant divisor to its divisor class in $\operatorname{Cl}(X_\Sigma)$. Furthermore, one has the following exact sequence

$$0 \to M \to \operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to 0$$

if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$.

Proof. Since the D_ρ are the irreducible components of $X_\Sigma \setminus T_N$, then by Theorem 4.1.2 one has the following exact sequence

$$\operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to \operatorname{Cl}(T_N) \to 0.$$

Note that $\mathbb{C}[x_1, \dots, x_n]$ is a UFD, the same is true for $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ since UFD is preserved under localization. This shows $\operatorname{Cl}(T_N) = 0$ since the coordinate ring of T_N is isomorphic to $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$. As a consequence, one has $\operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Div}(X_\Sigma)$ is surjective.

The composition $M \to \mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma$ is clearly zero since the first map maps m to the principal divisor $\mathrm{div}(\chi^m)$. Now suppose $D \in \mathrm{Div}_{T_N}(X_\Sigma)$ maps to 0 in $\mathrm{Cl}(X_\Sigma)$. Then $D = \mathrm{div}(f)$ for some $f \in \mathbb{C}(X_\Sigma)^*$. Since the support of D misses T_N, this implies $\mathrm{div}(f) = 0$ on T_N, and thus f is in fact a character $T_N \to \mathbb{C}^*$. Since all characters of T_N are of the form $c\chi^m$ for some $m \in M$, this shows

$$D = \mathrm{div}(f) = \mathrm{div}(c\chi^m) = \mathrm{div}(\chi^m)$$

as desired.

Finally, suppose $m \in M$ with $\mathrm{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} D_\rho$ is the zero divisor. Then $\langle m, u_\rho\rangle = 0$ for all $\rho \in \Sigma(1)$, which forces $m = 0$ if $\{u_\rho\}$ spans $N_\mathbb{R}$. This gives the desired exact sequence. Conversely it's also easy to see if the sequence is exact, then $\{u_\rho\}$ spans $N_\mathbb{R}$. □

Example 4.2.1. The fan of the blowup of $\mathbb{C}^2$ at the origin has ray generators $u_1 = e_1, u_2 = e_2$ and $u_3 = e_1 + e_2$, corresponding to the divisors D_1, D_2, D_0. Then the class group is generated by the classes of the D_i with the following relations

$$0 \sim \mathrm{div}(\chi^{e_1}) = D_1 + D_0$$
$$0 \sim \mathrm{div}(\chi^{e_2}) = D_2 + D_0.$$

Thus the class group is $\mathbb{Z}$ with generators $[D_1] = [D_2] = -[D_0]$.

Example 4.2.2. The fan of projective space $\mathbb{P}^n$ has ray generators given by $u_0 = -e_1 - \cdots - e_n$ and $u_1 = e_1, \ldots, u_n = e_n$. Thus the map $M \to \mathrm{Div}_{T_N}(\mathbb{P}^n)$ can be written as

$$\mathbb{Z}^n \to \mathbb{Z}^{n+1}$$
$$(a_1, \ldots, a_n) \mapsto (-a_1 - \cdots - a_n, a_1, \ldots, a_n).$$

Using the map

$$\mathbb{Z}^{n+1} \to \mathbb{Z}$$
$$(b_0, \ldots, b_n) \mapsto b_0 + \cdots + b_n,$$

one gets the exact sequence

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^{n+1} \to \mathbb{Z} \to 0$$

which proves $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$.

Example 4.2.3. The fan of Hirzebruch surface $\mathscr{H}_n$ has ray generators given by $u_1 = -e_1 + re_2, u_2 = e_2, u_3 = e_1, u_4 = -e_2$. The class group is generated by the classes of D_1, D_2, D_3, D_4 with relations

$$0 \sim \mathrm{div}(\chi^{e_1}) = -D_1 + D_3$$
$$0 \sim \mathrm{div}(\chi^{e_2}) = rD_1 + D_2 - D_4.$$

Thus $\mathrm{Cl}(\mathscr{H}_r) = \mathbb{Z} \times \mathbb{Z}$.

4.3. **The sheaf of a torus-invariant divisor.** Let D be a T_N-invariant divisor on a toric variety X_Σ. In this section we will give descriptions of the global sections $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$.

Proposition 4.3.1. If D is a T_N-invariant Weil divisor on X_Σ, then

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\operatorname{div}(\chi^m)+D\geq 0} \mathbb{C}\cdot\chi^m$$

Proof. Suppose $f \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$, then $\operatorname{div}(f)+D \geq 0$ implies $\operatorname{div}(f)|_{T_N} \geq 0$ since $D|_{T_N} = 0$. Then one has $f \in \mathbb{C}[M]$, that is,

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \subseteq \mathbb{C}[M].$$

Moreover, $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is T_N-invariant since D is T_N-invariant. By Theorem 1.2.2 one has

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\chi^m\in\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))} \mathbb{C}\cdot\chi^m.$$

Since $\chi^m \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ if and only if $\operatorname{div}(\chi^m) + D \geq 0$, as desired. □

Remark 4.3.1. For $D = \sum_\rho a_\rho D_\rho$ and $m \in M$, $\operatorname{div}(\chi^m)+D \geq 0$ is equivalent to

$$\langle m, u_\rho\rangle + a_\rho \geq 0$$

for all $\rho \in \Sigma(1)$. If we define

$$P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho\rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\},$$

then above proposition can be written as

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m\in P_D\cap M} \mathbb{C}\cdot\chi^m$$

Example 4.3.1. In Example 4.2.2 we show that $\operatorname{Cl}(\mathbb{P}^n) = \mathbb{Z}$, and thus divisors $D_1, D_2, \dots, D_n$ corresponding to the ray generators give the same sheaf, which is denoted by $\mathcal{O}_{\mathbb{P}^n}(1)$, and similarly the sheaves given by kD_i, where $k \in \mathbb{Z}$, is denoted by $\mathcal{O}_{\mathbb{P}^n}(k)$. For $D = kD_0$, a direct computation shows that

$$P_D = \begin{cases} \varnothing & k < 0 \\ k\Delta_n & k \geq 0, \end{cases}$$

where Δ_n is the standard n-simplex.

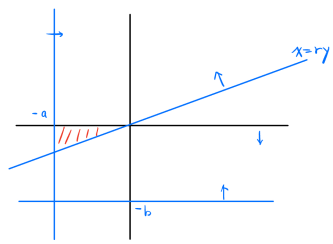
If we think characters as Laurent monomials $t^m = t_1^{a_1}\dots t_n^{a_n}$, where $m = (a_1, \dots, a_n)$. It follows that

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \cong \{f \in \mathbb{C}[t_1, \dots, t_n] \mid \deg(f) \leq k\}.$$

By considering the homogenization of such a polynomial, one has

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \cong \{f \in \mathbb{C}[x_0, \dots, x_n] \mid f \text{ is homogenous with } \deg(f) \leq k\}.$$

Example 4.3.2. In this example we compute the effective cone of Hirzebruch surface $\mathscr{H}_r$. Recall Example 4.2.3 implies that the classes of $\{D_2, D_3\}$ gives a basis of $\mathrm{Cl}(\mathscr{H}_2) \cong \mathbb{Z}^2$. For divisor $D = aD_3 + bD_2$, the picture of P_D is given by



This shows $P_D \cap M \neq \varnothing$ if and only if $a > 0$ and $b > 0$, and thus the effective cone of $\mathscr{H}_r$ looks like