Note on Projective Toric Variety

ZCC
zcc22@mails.tsinghua.edu.cn

September 30, 2023

Contents

Chapter 1

Preliminaries

1.1 Things about Projective Geometry

We assume the reader has know basic knowledge of projective geometry, e.g. [1, pp8-14], [2, pp49-54].

1.1.1 Some Basic Constructions

Definition 1.1.1 (Veronese embedding)**.** Fix $n, d > 0$, let $M_0, \cdots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \cdots, x_n$, where $N = \binom{n+d}{n} - 1$. We have a map

$$\begin{aligned} v_d \colon \mathbb{P}^n &\to \mathbb{P}^N \\ a &\mapsto [M_0(a), \cdots, M_N(a)] \end{aligned}$$

which defines a closed immersion of varieties. We call v_d the *d-uple embedding* or *d-th Veronese embedding* of $\mathbb{P}^n$.

When $n = 1$, $v_d(\mathbb{P}^1)$ is a nonsingular curve in $\mathbb{P}^d$, which is called the *rational normal curve of degree d.* When $n = 2$, $v_d(\mathbb{P}^2)$ is a nonsingular surface in $\mathbb{P}^{\frac{d(d+3)}{2}}$, which is called the *Veronese surface of degree d.*

Definition 1.1.2 (Segre embedding)**.** The map

$$\begin{aligned} \sigma \colon \mathbb{P}^m \times \mathbb{P}^n &\to \mathbb{P}^{mn+m+n} \\ (a, b) &\mapsto [\cdots a_i b_j]_{0 \le i \le m, 0 \le j \le n} \end{aligned}$$

defines a closed immersion of varieties. We call σ the *Segre embedding*.

When $m = n = 1$, $\mathbb{P}^1 \times \mathbb{P}^1$ is a quatric surface in $\mathbb{P}^3$, which is called the *Segre surface*.

Definition 1.1.3 (weighted projective space)**.** Let $S = \mathbb{C}[x_0, \cdots, x_n]$ be the graded polynomial ring with $\deg x_i = d_i$, then

$$\mathbb{P}(d_0, \cdots, d_n) := \operatorname{Proj} S,$$

is called a *weighted projective space*.

We may always assume any n elements in $\{d_0, \cdots, d_n\}$ have no common factor (why?), in which case such weighted projective space is well-formed.

Definition 1.1.4 (Grassmannian)**.** The Grassmannian manifold (over $\mathbb{C}$) $\operatorname{Gr}(k, n)$, consists of linear subspace of dimension k of a given linear space of dimension $n \geq k$, is a projective variety via the morphism

$$\begin{aligned} \operatorname{Gr}(k, n) &\to \operatorname{Gr}\left(1, \binom{n}{k}\right) = \mathbb{P}^{\binom{n}{k}-1} \\ W &\mapsto \det W \end{aligned}$$

which is called a *Grassmannian variety*.

1.1.2 Affine Cone and Projective Normality

Let X be a projective variety in $\mathbb{P}^n$, and $\mathbf{I}(V)$ be the ideal of $\mathbb{C}[x_0, \cdots, x_n]$ generated by homogeneous polynomials vanishing on X, we say

$$S(X) := \mathbb{C}[x_0, \cdots, x_n]/\mathbf{I}(V),$$

is the *homogeneous coordinate ring* of X (with respect to the embedding $X \subset \mathbb{P}^n$). For following discussion, we fix the notation as above.

Definition 1.1.5 (affine cone)**.** There is an affine variety $\widehat{X} \subset \mathbb{A}^{n+1}$ corresponds to $\mathbf{I}(X)$. We say $\widehat{X}$ is the *affine cone* of X. See exercise I.2.10 in [1, p12].

Definition 1.1.6 (projective normality)**.** We say $X \subset \mathbb{P}^n$ is *projectively normal* if its affine cone $\widehat{X}$ is a normal variety, i.e. $S(X)$ is a normal ring.

Theorem 1.1.7. $X \subset \mathbb{P}^n$ is projectively normal, iff X is normal (as a variety) and the natural maps

$$\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(r)) \to \Gamma(X, \mathscr{O}_X(r)),$$

are surjective for all $r \geq 0$.

Proof. Omitted. This is exercise II.5.14(d) in [1, p126], which generalizes exercise I.3.18 in [1, p23] and exercise 2.1.5 in [2, p62]. □

Remark 1.1.8. There is a normal projective variety which is not projectively normal, see example 2.1.10 in [2, p61]. More generally, a nonsingular curve of type (a, b) on the Segre surface $xw = yz$ in $\mathbb{P}^3$ is projectively normal iff $|a - b| \leq 1$, see exercise III.5.6 in [1, p231].

Example 1.1.9 (rational normality)**.** The images of Veronese embedding and Segre embedding are both projectively normal in corresponding spaces. They are both *rational normal*, where "rational" means they are birational to projective space and "normal" means they are projectively normal. It is interesting that

- A non-degenerate irreducible curve in $\mathbb{P}^n$ is rational normal iff it is of degree n, iff it is the Veronese curve.
- A non-degenerate ruled surface in $\mathbb{P}^n$ is rational normal, then it is of degree $n-1$; conversely, a non-degenerate irreducible surface of degree $n-1$ in $\mathbb{P}^n$ is either a rational normal scroll or the Veronese surface.

1.2 Things about Combination Theory

We need some combinational concepts.

1.2.1 Polytopes

See [2, pp63-67] for details.

Definition 1.2.1 (cone,affine hull, convex hull)**.** Suppose V is an $\mathbb{R}$-linear space of finite dimension and $S \subset V$ is a finite subset, then define its

- *cone* by
$$\mathrm{cone}(S) = \left\{ \sum_{u \in S} \lambda_u u : \lambda_u \geq 0 \right\},$$
- *affine hull* by
$$\mathrm{aff}(S) = \left\{ \sum_{u \in S} \lambda_u u : \sum_{u \in S} \lambda_u = 1 \right\},$$
- and *convex hull* by $\mathrm{conv}(S) = \mathrm{cone}(S) \cap \mathrm{aff}(S)$.

A *polytope* in V is a convex hull of some finite subset, and there is a reasonable way to define the *faces* and *dimension* of a polytope. A face of a polytope of dimension n is a facet(resp. edge, vertex) if it has dimension $n-1$(resp. $1, 0$). One can also define the (Minkowsiki) sum, multiple and dual of a polytope.

Let M be a lattice with dual N, we may always consider the space $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ with dual $N_{\mathbb{R}}$. A *lattice polytope* in $M_{\mathbb{R}}$ is the convex hull of a finite subset of M.

Example 1.2.2 (Birkhoff polytope)**.** Consider the case $M = \mathbb{Z}^{d \times d}$. A matrix $A \in \mathbb{R}^{d \times d}$ is *doubly-stochastic* if it has nonnegative entries and its row and column sums are all 1. Then the set $\mathscr{M}_d$ of all doubly-stochastic matrices form a lattice polytope, of vertices all permutation matrices. See example 2.2.2 and 2.2.5 in [2, p64,p66].

Definition 1.2.3 (combinational equivalence)**.** Two polytopes P_1, P_2 are *combinationally equivalent* if there is abijection

$$\{\text{faces of } P_1\} \simeq \{\text{faces of } P_2\},$$

which preserves dimensions, intersections, and the face relation.

1.2.2 Normal Polytopes

The concepts of normality and ampleness of a polytope raises from the normality of the associated projective toric variety.

Definition 1.2.4 (normal polytope)**.** A lattice polytope $P \subset M_{\mathbb{R}}$ is *normal* if

$$(kP) \cap M + (lP) \cap M = ((k + k)P) \cap M,$$

for all $k, l \in \mathbb{N}$.

Example 1.2.5 (1-dimensional polytope)**.** An 1-dimensional polytope must be a (single closed) line segment, and it is a lattice polytope iff its two endpoints are lattice points. Let L be such a line segment in $M_{\mathbb{R}}$ joint two lattice points m_1, m_2, then kL is the line segment joint km_1, km_2, which is just the k-uple sum $L + L + \cdots + L$. This shows that L is normal.

Theorem 1.2.6. If $P \subset M_{\mathbb{R}}$ is a full dimensional lattice polytope of dimension $n \geq 2$, then kP is normal for all $k \geq n - 1$.

Proof. See theorem 2.2.12 in [2, p69]. □

Remark 1.2.7. The bound $k \geq n - 1$ can be strengthen, see lemma 2.2.16 in [2, p71].

Corollary 1.2.8. All lattice polygon in $\mathbb{R}^2$ is normal.

1.2.3 Very Ample Polytopes

In algebraic geometry, the very ampleness of a sheaf, roughly speaking, means that it has enough sections. As for polytopes, the very-ampleness means it has enough lattice points.

Definition 1.2.9 (very-ampleness)**.** A lattice polytope $P \subset M_{\mathbb{R}}$ is *very ample* if for every vertex $m \in P$ the semigroup $\mathsf{S}_{P,m}$ generated by

$$P \cap M - m := \{m' - m : m' \in P \cap M\},$$

is *saturated* in M, i.e. $kp \in \mathsf{S}_{P,m}$ implies $p \in \mathsf{S}_{P,m}$ for all positive integer k and lattice point $p \in M$.

Theorem 1.2.10. A normal lattice polytope is very ample.

Proof. See proposition 2.2.18 in [2, p71]. □

1.2.4 Fans

See [2, p106]. We do not need the general terminology "fan", which will be used in the construction of toric abstract variety. In the construction of projective toric variety, we will use *normal fans* of a lattice polytope, see [2, pp76-83].

Chapter 2

Constructions of Projective Toric Variety

2.1 Using Lattice Points

We first construct various examples of projective toric variety by an algebraic way. The main reference is [2, pp54-62]

2.1.1 Torus of Projective Space

It is known that the affine space $\mathbb{C}^n$ has a standard torus $(\mathbb{C}^*)^n$, making it a toric variety. As for the projective space $\mathbb{P}^n$, there is a nonempty open subset defined by $x_0 \cdots x_n \neq 0$, which is

$$T_{\mathbb{P}^n} := \{[t_0, \cdots, t_n] \in \mathbb{P}^n : t_i \in \mathbb{C}^*\}.$$

Theorem 2.1.1. $\mathbb{P}^n$ is a toric variety.

Proof. We have the isomorphism

$$\begin{aligned} T_{\mathbb{P}^n} &\to (\mathbb{C}^*)^n \\ [t_0, \cdots, t_n] &\to (t_1/t_0, \cdots, t_n/t_0) \end{aligned}$$

so $T_{\mathbb{P}^n}$ is indeed a torus. The multiplication of $T_{\mathbb{P}^n}$ is defined by

$$[s_0, \cdots, s_n] \cdot [t_0, \cdots, t_n] = [s_0t_0, \cdots, s_nt_n],$$

which obviously extends an algebraic action $T_{\mathbb{P}^n} \times \mathbb{P}^n \to \mathbb{P}^n$. So $\mathbb{P}^n$ is a toric variety. □

Note that $T_{\mathbb{P}^n}$ can be identified with the quotient group $(\mathbb{C}^*)^{n+1}/\Delta(\mathbb{C}^*)$, where

$$\begin{aligned}\Delta\colon \mathbb{C}^* &\to (\mathbb{C}^*)^{n+1}\\ t &\mapsto (t, t, \cdots, t)\end{aligned}$$

is the diagonal morphism, it has character lattice

$$\mathscr{M}_n = \{(a_0, \cdots, a_n) \in \mathbb{Z}^{n+1} : \sum a_i = 0\},$$

and lattice of one-parameter subgroups

$$\mathscr{N}_n = \mathbb{Z}^{n+1}/\mathbb{Z}(1, \cdots, 1).$$

Lemma 2.1.2. Suppose V is a (locally closed) subvariety of $\mathbb{P}^n$ containing $T_{\mathbb{P}^n}$, and is toric under the natural action of $T_{\mathbb{P}^n}$ (e.g. the affine piece $x_0 \neq 0$). If T is an irreducible closed subgroup of $T_{\mathbb{P}^n}$, then it is a torus and its Zariski closure in V is a toric variety.

Proof. By proposition 1.1.1(b) in [2, p11], T is indeed a torus. Denote by $\overline{T}$ for the closure of T in V, then since T is closed in $T_{\mathbb{P}^n}$ we have

$$T = \overline{T} \cap T_{\mathbb{P}^n}.$$

Since $T_{\mathbb{P}^n}$ is open in $\mathbb{P}^n$, it is also open in $\overline{T}$. For each $t \in T$, its action on V is well-defined, and takes closed subvarieties to closed subvarieties (because it is an automorphism of V with the inverse defined by the action by t^{-1}). Thus

$$T = t \cdot T \subset t \cdot \overline{T},$$

i.e. $t \cdot \overline{T}$ is a closed subvariety contains T. So we have $\overline{T} \subset t \cdot \overline{T}$. Replace t by t^{-1}, it then follows that $\overline{T} = t \cdot \overline{T}$, i.e. the action of T on itself extends to its closure $\overline{T}$. So $\overline{T}$ is toric. □

2.1.2 The Construction of $X_{\mathscr{A}}$

Let T_N be a torus with character lattice M and $\mathscr{A} = \{m_1, \cdots, m_s\}$ be a finite subset of M, then there is a morphism

$$\begin{aligned} \Phi_{\mathscr{A}} \colon T_N &\to (\mathbb{C}^*)^s \\ t &\mapsto (\chi^{m_1}(t), \cdots, \chi^{m_s}(t)) \end{aligned}$$

and a natural projection

$$\pi \colon (\mathbb{C}^*)^s \to T_{\mathbb{P}^{s-1}}.$$

Lemma 2.1.3. The image of $\Phi_{\mathscr{A}}$ is a torus in $(\mathbb{C}^*)^s \subset \mathbb{C}^s$, and its Zariski closure $Y_{\mathscr{A}}$ in $\mathbb{C}^s$ is toric. The image of $\pi \circ \Phi_{\mathscr{A}}$ is also a torus $T_{\mathscr{A}}$ in $T_{\mathbb{P}^{s-1}} \subset \mathbb{P}^{s-1}$, and its Zariski closure $X_{\mathscr{A}}$ in $\mathbb{P}^{s-1}$ is toric.

Proof. Just combine proposition 1.1.1(a) in [2, p11] and lemma 2.1.2. □

Definition 2.1.4. $Y_{\mathscr{A}}$ is called the *affine toric variety associated to* $\mathscr{A}$, and $X_{\mathscr{A}}$ is called the *projective toric variety associated to* $\mathscr{A}$.

Example 2.1.5 (cuspidal cubic curve)**.** Let $T_N = \mathbb{C}^*$ then $M = \mathbb{Z}$. Consider $\mathscr{A} = \{3, 4, 1\}$, then the corresponding map $\pi \circ \Phi_{\mathscr{A}}$ is defined by

$$t \mapsto (t^3, t^4, t) \mapsto [t^2, t^3, 1].$$

One can see that $Y_{\mathscr{A}}$ is the affine cubic curve in $\mathbb{C}^3$ defined by $x = z^3, y = z^4$, and $X_{\mathscr{A}}$ is the cuspidal cubic curve defined by $x^3 = y^2 z$ and the torus $T_{\mathscr{A}}$ is just the complement of the cusp $[0, 0, 1]$.

Example 2.1.6. Let T_N be the space of all $n \times n$ matrices with nonzero entries, then $M = \mathbb{Z}^{n \times n}$. Consider

$$\mathscr{A} = \mathscr{P}_n = \{n \times n \text{ permutation matrices}\} \subset M,$$

then the corresponding map $\pi \circ \Phi_{\mathscr{A}}$ is defined by

$$\begin{aligned} T_N &\to \mathbb{P}^{n!-1} \\ (t_{ij})_{1 \le i,j \le n} &\mapsto \left[\prod_{i=1}^{n} t_{i\sigma(i)} \right]_{\sigma \in S_n} \end{aligned}$$

One can see that

$$\prod_{\sigma \text{ even}} x_\sigma - \prod_{\sigma \text{ odd}} x_\sigma \in \mathbf{I}(X_{\mathscr{A}}),$$

but it is not easy to determine $\mathbf{I}(X_{\mathscr{A}})$ and the cooresponding torus.

2.1.3 Basic Properties of $X_{\mathscr{A}}$

In general, $Y_{\mathscr{A}} \neq \widehat{X}_{\mathscr{A}}$, and example 2.1.5 shows that they can even have the same dimension.

Theorem 2.1.7. $Y_{\mathscr{A}} = \widehat{X}_{\mathscr{A}}$ iff the ideal

$$I_{\mathscr{A}} = \langle \prod x_i^{a_i} - \prod x_i^{b_i} \Big| a_i, b_i \in \mathbb{N}, \sum (a_i - b_i) m_i = 0 \rangle,$$

is homogeneous in $\mathbb{C}[x_0, \cdots, x_n]$, iff there is a one parameter subgroup u and positive integer k such that $\langle m_i, u \rangle = k$ for $i = 1, \cdots, s$.

Proof. See [2, p56]. □

Another task is to determine the dimension of $X_{\mathscr{A}}$. Consider the subgroup of $\mathbb{Z}\mathscr{A}$ defined by

$$\mathbb{Z}'\mathscr{A} = \left\{ \sum a_i m_i \mid a_i \in \mathbb{Z}, \sum a_i = 0 \right\},$$

then

Theorem 2.1.8. $\mathbb{Z}'\mathscr{A}$ is the character lattice of $X_{\mathscr{A}}$, and its rank is the dimension of the samllest affine subspace of $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ containing $\mathscr{A}$.

Proof. Omitted in this note, I will explain offline. □

Corollary 2.1.9. The following numbers equal:

(1) $\dim X_{\mathscr{A}}$,

(2) $\dim T_{\mathscr{A}}$,

(3) dimension of the samllest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$,

(4) $\operatorname{rank} \mathbb{Z}'\mathscr{A}$,

(5) $\operatorname{rank} \mathbb{Z}\mathscr{A} - \epsilon$, where

$$\epsilon = \begin{cases} 1, & Y_{\mathscr{A}} = \widehat{X}_{\mathscr{A}}, \\ 0, & \text{otherwise}. \end{cases}$$

Example 2.1.10. Recall example 2.1.6. It is known that the permutation metrices span a linear subspace of $\mathbb{C}^{n\times n}$ of dimension $(n-1)^2-1$. Then one can easily see that $\mathbb{Z}\mathscr{P}_n$ is of rank $(n-1)^2-1$. One can also easily see that $I_{\mathscr{P}_n}$ is homogeneous, so $\dim X_{\mathscr{P}_n} = (n-1)^2-1$. In particular, if $n=3$, then $X_{\mathscr{P}_3}$ is a hypersurface in $\mathbb{P}^4$, so $X_{\mathscr{P}_3}$ is just defined by $x_{123}x_{231}x_{312} = x_{132}x_{213}x_{321}$.

Now suppose U_i is the i-th affine piece of $\mathbb{P}^{s-1}$ defined by $x_i \neq 0$, then

$$T_{\mathscr{A}} = X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}} \subset X_{\mathscr{A}} \cap U_i.$$

This shows that $X_{\mathscr{A}} \cap U_i$ is the Zariski closure of $T_{\mathscr{A}}$ in $U_i \simeq \mathbb{C}^{s-1}$ and is an affine toric variety.

Theorem 2.1.11. Let

$$\mathscr{A}_i = \mathscr{A} - m_i = \{m_j - m_i : j \neq i\},$$

then there is an isomorphism $X_{\mathscr{A}} \cap U_i \simeq Y_{\mathscr{A}_i}$.

Proof. Easy. □

2.2 Using Polytopes

The construction by lattice points also determine a specified embedding in some projective space, and we want an intrinsic way to construct projective toic variety.

2.2.1 The Very Ample Case

Lemma 2.2.1. Let $\mathscr{A} = \{m_1, \cdots, m_s\} \subset M$ be a set of lattice points and $P = \mathrm{conv}(\mathscr{A}) \subset M_{\mathbb{R}}$ be its convex hull. Then all vertices of P lie in $\mathscr{A}$. Moreover, consider

$$J = \{j : m_j \text{ is a vertex of } P\},$$

then $X_{\mathscr{A}} = \bigcup_{j\in J}(X_{\mathscr{A}} \cap U_j)$, where U_j is the j-th affine piece of $\mathbb{P}^{s-1}$.

Proof. For each i the lattice point m_i lies in $P \cap M$, so there exsit $k, k_j \in \mathbb{N}$ such that

$$km_i = \sum_{j \in J} k_j m_j, \text{ and } k = \sum_{j \in J} k_j.$$

There is a k_{j_0} which is nonzero, hence

$$m_i - m_{j_0} = (k_{j_0} - 1)(m_{j_0} - m_i) + \sum_{j \in J, j \neq j_0} k_j(m_j - m_i),$$

lie in the smeigroup generated by $\mathscr{A}_i = \mathscr{A} - m_i$ and is the inverse of $m_{j_0} - m_i$. It follows that

$$X_{\mathscr{A}} \cap U_i \cap U_{j_0} = X_{\mathscr{A}} \cap U_i,$$

hence $X_{\mathscr{A}} \cap U_i \subset X_{\mathscr{A}} \cap U_{j_0}$. So any affine piece of $X_{\mathscr{A}}$ in 2.1.11 is contained in some affine piece determined by a vertex of P. □

Now suppose P is a full dimensional very ample lattice polytope in M, of dimension n. Then $P \cap M = \{m_1, \cdots, m_s\}$ are a finite set of lattice points and we can define the projective toric variety $X_{P \cap M}$ associated to it. By above lemma 2.2.1, we have

$$X_{P \cap M} = \bigcup_{m_i \text{ vertex of } P} (X_{P \cap M} \cap U_i).$$

Lemma 2.2.2. For each vertex m_i of P the corresponding affine piece $X_{P \cap M} \cap U_i$ is just the affine toric variety associated to the strongly convex rational cone σ_i dual to the cone

$$\text{cone}(P \cap M - m_i),$$

with character lattice M. As a corollary, such $X_{P \cap M}$ is a normal variety.

Proof. Easy. See [2, p75]. □

2.2.2 The Construction of X_P

We may first throw out the definition.

Definition 2.2.3. Let P be a full dimensional lattice polytope, and k be a positive integer such that kP is very ample, then

$$X_P = X_{(kP)\cap M},$$

is called the *projective toric variety associated to* P.

By theorems 1.2.6 and 1.2.10, such integer k exsits (and not unique) and for any two integers k, l satisfy this condition there exists an isomorphism $X_{(kP)\cap M} \simeq X_{(lP)\cap M}$. Thus X_P is well-defined.

Example 2.2.4 (Veronese embedding)**.** Consider the standard n-simplex $\Delta_n = \text{conv}(0, e_1, \cdots, e_n)$ in $\mathbb{R}^n$, one can see that

$$k\Delta_n = \text{conv}(0, ke_1, \cdots, ke_n),$$

and $k\Delta_n \cap \mathbb{Z}^n$ has $N = \binom{k+n}{k}$ lattice points. Moreover, $k\Delta_n$ is a full-dimensional normal polytope, and the variety $X_{(k\Delta_n)\cap\mathbb{Z}^n}$ is just the image of the Veronese embedding

$$v_k \colon \mathbb{P}^n \mapsto \mathbb{P}^{N-1}.$$

Thus X_{Δ_n} is just $\mathbb{P}^n$.

By definition, and theorem 1.3.5 in [2, p37], one can see that

Theorem 2.2.5. Let P be a full dimensional lattice polytope. Then

(a) X_P is a normal projective toric variety.

(b) X_P is projectively normal under the embedding given by kP iff kP is a normal polytope.

Bibliography

[1] Robin Hartshorne. (1977) *Algebraic Geometry*, Springer-Verlag, New York.

[2] David A. Cox, John B. Little, Henry K. Schenck. (2011) *Toric Variety*, American Mathematical Society.