Part 1. Basic theories of toric varieties

1. Preliminaries

1.1. Torus.

Definition 1.1.1 (torus)**.** A torus T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from the isomorphism.

Definition 1.1.2 (character)**.** A character of a torus T is a morphism $\chi\colon T \to \mathbb{C}^*$ that is a group homomorphism.

Definition 1.1.3 (one-parameter subgroup)**.** A one-parameter subgroup of a torus T is a morphism $\lambda\colon\ \mathbb{C}^* \to T$ that is a group homomorphism.

Example 1.1.1. All characters of $(\mathbb{C}^*)^n$ arise from

$$\chi^{(a_1,\dots,a_n)}\colon (t_1,\dots,t_n) \mapsto t_1^{a_1}\dots t_n^{a_n},$$

and all one-parameter subgroups of $(\mathbb{C}^*)^n$ arise from

$$\lambda^{(b_1,\dots,b_n)}\colon t \mapsto (t^{b_1},\dots,t^{b_n}),$$

where $(a_1,\dots,a_n),(b_1,\dots,b_n) \in \mathbb{Z}^n$.

1.2. Affine semigroups.

Definition 1.2.1 (affine semigroup)**.** An affine semigroup S is a semigroup group such that

(1) The binary operation on S is commutative.
(2) The semigroup is finitely generated.
(3) The semigroup can be embedded in a lattice M.

Definition 1.2.2 (saturated)**.** An affine semigroup $S \subseteq M$ is saturated if for all $0 \neq k \in \mathbb{N}$ and $m \in M$, $km \in S$ implies $m \in S$.

Example 1.2.1. $\mathbb{N}^n \subseteq \mathbb{Z}^n$ is an affine semigroup.

Example 1.2.2. Given a finite set $\mathscr{A}$ of a lattice M, $\mathbb{N}\mathscr{A} \subseteq M$ is an affine semigroup.

Definition 1.2.3 (semigroup algebra)**.** Let $S \subseteq M$ be an affine semigroup. The semigroup algebra $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication is induced by the semigroup structure.

Remark 1.2.1*.* To make this precise, we write

$$\mathbb{C}[S] = \{\sum_{m\in S} c_m\chi^m \mid c_m \in C \text{ and } c_m = 0 \text{ for all but finitely many } m\}$$

with multiplication given by

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$

If $S = \mathbb{N}\mathscr{A}$ for $\mathscr{A} = \{m_1,\dots,m_s\}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1},\dots,\chi^{m_s}]$.

Example 1.2.3. The affine semigroup $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives the polynomial ring

$$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \dots, x_n]$$

where $x_i = \chi^{e_i}$ and $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{Z}^n$.

Example 1.2.4. If $e_1, \dots, e_n$ is a basis of a lattice M, then M is generated by $\mathscr{A} = \{\pm e_1, \dots, \pm e_n\}$ as an affine semigroup, and the semigroup algebra gives the Laurent polynomial ring

$$\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

where $x_i = \chi^{e_i}$.

Theorem 1.2.1. Let T_N be a n-torus with group M consisting of characters and group N consisting of one-parameter subgroups. Then

(1) M, N are lattices of rank n.
(2) M, N are dual lattices, that is $N \cong \operatorname{Hom}(M, \mathbb{Z})$ and $N \cong \operatorname{Hom}(N, \mathbb{Z})$.
(3) $T_N \cong \operatorname{Spec} \mathbb{C}[M]$ as varieties.
(4) $T_N \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \operatorname{Hom}(M, \mathbb{C}^*)$ as groups.

For torus T_N with character group M, there is a natural action of T_N on the semigroup algebra $\mathbb{C}[M]$ as follows: For $t \in T_N$ and $\chi^m \in M$, $t \cdot \chi^m$ is defined by $p \mapsto \chi^m(t^{-1}p)$ for $p \in T_N$.

Theorem 1.2.2. Let $A \subseteq \mathbb{C}[M]$ be a subspace stable under the action of T_N. Then

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

Proof. See Lemma 1.1.16 in [CLS11]. □

1.3. **Strongly convex rational polyhedral cones.** From now on, unless otherwise specified, we always assume M, N are dual lattices with associated $\mathbb{R}$-vector spaces $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and the pairing between M and N is denoted by $\langle \text{-}, \text{-} \rangle$.

1.3.1. *Convex polyhedral cones.*

Definition 1.3.1 (convex polyhedral cone)**.** Let $S \subseteq N_{\mathbb{R}}$ be a finite subset. A convex polyhedral cone in $N_{\mathbb{R}}$ generated by S is a set of the form

$$\sigma = \operatorname{Cone} S = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\} \subseteq N_{\mathbb{R}}.$$

Notation 1.3.1. $\operatorname{Cone}(\varnothing) = \{0\}$.

Remark 1.3.1. A convex polyhedral cone is convex, that is $x, y \in \sigma$ implies $\lambda x + (1 - \lambda) y \in \sigma$ for all $0 \leq \sigma \leq 1$, and it's a cone, that is $x \in \sigma$ implies $\lambda x \in \sigma$ for all $\lambda \geq 0$. Since we will only consider convex cones, the cones satisfying Definition 1.3.1 will be called polyhedral cone for convenience.

Definition 1.3.2 (dimension)**.** The dimension of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is the dimension of the smallest subspace $W \subseteq N_{\mathbb{R}}$ containing σ, and such W is called the span of σ.

Definition 1.3.3 (dual cone)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral. The dual cone is defined by

$$\sigma^{\vee} := \{u \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Definition 1.3.4 (hyperplane)**.** Given $m \in M_{\mathbb{R}}$, the hyperplane given by m is defined by

$$H_m := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}},$$

and the closed half-space given by m is defined by

$$H_m^+ := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

Definition 1.3.5 (supporting hyperplane)**.** The supporting hyperplane of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is a hyperplane H_m such that $\sigma \subseteq H_m^+$, and H_m^+ is called a supporting half-space.

Remark 1.3.2. H_m is a supporting hyperplane of σ if and only if $m \in \sigma^{\vee}$, and if $m_1, \ldots, m_s$ generates $\sigma^{\vee}$, then

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+.$$

Thus every polyhedral cone is an intersection of finitely many closed half-spaces.

Definition 1.3.6 (face)**.** A face of a polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^{\vee}$, written $\tau \preceq \sigma$. Faces $\tau \neq \sigma$ are called proper faces, written $\tau \prec \sigma$.

Definition 1.3.7 (facet and edge)**.** A facet of a polyhedral cone σ is a face of codimension one, and an edge of σ is a face of dimension one.

Theorem 1.3.1. Suppose σ is a polyhedral cone. Then

(1) Every face of σ is a polyhedral cone.
(2) An intersection of two faces of σ is again a face of σ.
(3) A face of a face of σ is again a face of σ.
(4) If $\tau \preceq \sigma$, $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.
(5) Every face of $\sigma^{\vee}$ can be uniquely written as $\sigma^{\vee} \cap \tau^{\perp}$, where $\tau \preceq \sigma$ and

$$\tau^{\perp} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = 0, \forall u \in \tau\}$$

1.3.2. *Strongly convex.*

Definition 1.3.8 (strongly convex)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex if $\{0\}$ is a face of σ.

Theorem 1.3.2. Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then the following statements are equivalent:

(1) σ is strongly convex.

(2) $\{0\}$ is a face of σ.
(3) σ contains no positive-dimensional subspace of $N_{\mathbb{R}}$.
(4) $\sigma \cap (-\sigma) = \{0\}$.
(5) $\dim \sigma^\vee = n$.

1.3.3. *Rational polyhedral cones.*

Definition 1.3.9 (rational)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is rational if $\sigma = \mathrm{Cone}(S)$ for some finite subset $S \subseteq N$.

Definition 1.3.10 (ray generator)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and ρ be an edge of σ. The unique generator of semigroup $\rho \cap N$ is called ray generator of ρ, written u_ρ.

Remark 1.3.3. The ray generator is well-defined: Since σ is strongly convex, one has edge of σ is a ray as $\{0\}$ is its face, and since σ is rational, the semigroup $\rho \cap N$ is generated by a unique element, otherwise contradicts to the fact ρ is an edge, that is it's of dimension one.

Lemma 1.1. A strongly convex rational polyhedral cone is generated by the ray generators of its edges.

1.3.4. *Other properties.*

Definition 1.3.11 (smooth and simplicial)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.

(1) σ is smooth if its ray generators form part of a $\mathbb{Z}$-basis of N.
(2) σ is simplical if its ray generators are linearly independent over $\mathbb{R}$.

2. Toric variety

2.1. Cones and affine toric varieties.

2.1.1. *Construction.*

Definition 2.1.1 (affine toric variety)**.** An affine toric variety is an irreducible affine variety V containing a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on V.

Proposition 2.1.1 (Gordan's lemma)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone. The semigroup $S_\sigma := \sigma^\vee \cap M$ is finitely generated.

Proof. See Proposition 1.2.17 in [CLS11]. □

Theorem 2.1.1. Let $\sigma \subsetneq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then

$$U_\sigma := \operatorname{Spec}(\mathbb{C}[S_\sigma])$$

is a normal affine toric variety with torus $T_N \cong \operatorname{Spec}\mathbb{C}[M]$.

Proof. If $\sigma \subsetneq N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone, then by Proposition 2.1.1 one has S_σ is finitely generated. Suppose $\mathscr{A} = \{m_1, \dots, m_s\}$ is a generator of S_σ. Then the strongly convexity implies $\mathbb{Z}\mathscr{A} = M$. If we define $T_N = \operatorname{Spec}\mathbb{C}[M]$, then M and N can be viewed as characters and one one-parameter subgroups of T_N respectively. Consider

$$\begin{aligned} \Phi_{\mathscr{A}} \colon T_N &\to (\mathbb{C}^*)^s \\ t &\mapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t)). \end{aligned}$$

It's clear to see $\Phi_{\mathscr{A}}$ gives a closed immersion from T_N to $(\mathbb{C}^*)^s$ by checking the induced morphism on coordinate rings. If we use T to denote the image of T_N in $(\mathbb{C}^*)^s$ and use $Y_{\mathscr{A}}$ to denote the Zariski closure of T in $\mathbb{C}^s$, then $Y_{\mathscr{A}} \cap (\mathbb{C}^*)^s = T$. Moreover, T is irreducible since it's a torus, so the same is true for its Zariski closure $Y_{\mathscr{A}}$. Consider the morphism on coordinate rings corresponding to $\Phi_{\mathscr{A}} \colon T_N \to \mathbb{C}^s$

$$\begin{aligned} \Phi^\sharp_{\mathscr{A}} \colon \mathbb{C}[x_1, \dots, x_s] &\to \mathbb{C}[M] \\ x_i &\mapsto \chi^{m_i}. \end{aligned}$$

Since $Y_{\mathscr{A}}$ is the Zariski closure of T, the coordinate ring of $Y_{\mathscr{A}}$ is given by

$$\mathbb{C}[x_1, \dots, x_n]/\ker\Phi^\sharp_{\mathscr{A}} = \operatorname{im}\Phi^\sharp_{\mathscr{A}} = \mathbb{C}[S_\sigma].$$

Thus $Y_{\mathscr{A}} \cong U_\sigma \cong \operatorname{Spec}\mathbb{C}[S_\sigma]$.

To see U_σ is normal, it suffices to show $\mathbb{C}[S_\sigma]$ is integrally closed. Suppose $\rho_1, \dots, \rho_r$ are rays of σ. Then by Lemma 1.1 one has

$$\sigma^\vee = \bigcap_{i=1}^r \rho_i^\vee.$$

Intersecting with M gives $S_\sigma = \bigcup_{i=1}^r S_{\rho_i}$, which easily implies

$$\mathbb{C}[S_\sigma] = \bigcap_{i=1}^r \mathbb{C}[S_{\rho_i}].$$

Thus it suffices to show each strongly convex rational cone ρ of dimension one, $\mathbb{C}[S_\rho$ is integrally closed. Suppose u_ρ is the ray generators of ρ, and extends u_ρ to a basis of N as $e_1 = u_\rho, e_2, \dots, e_n$ with dual basis $x_1, \dots, x_n$ in M. Then

$$\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm}, \dots, x_n^{\pm}].$$

It's clear $\mathbb{C}[S_\rho]$ is integrally closed. □

Remark 2.1.1. In fact, for any normal affine toric variety X, there exists a strongly convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$ such that $X \cong U_\sigma$.

Remark 2.1.2 (lattice point construction). More generally, given a torus T_N with character lattice M, a set $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M$ gives the characters $\chi^{m_i} \colon T_N \to \mathbb{C}^*$, and thus the following map

$$\begin{aligned} \Phi_\mathscr{A} \colon T_N &\to \mathbb{C}^s \\ t &\mapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t)). \end{aligned}$$

By the same argument, one can see the Zariski closure of the image of the map $\Phi_\mathscr{A}$, denoted by $Y_\mathscr{A}$, is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathscr{A}$. Moreover, $Y_\mathscr{A}$ is normal if and only if $\mathscr{A}$ is the set of generators of S_σ for some strongly convex rational polyhedral cone $\sigma \subseteq \mathbb{N}_\mathbb{R}$.

Remark 2.1.3 (affine semigroup construction). Let $S \subseteq M$ be an affine semigroup with generators $\mathscr{A} = \{m_1, \dots, m_s\}$. Then consider

$$\begin{aligned} \Phi^\sharp_\mathscr{A} \colon \mathbb{C}[x_1, \dots, x_s] &\to \mathbb{C}[M] \\ x_i &\mapsto \chi^{m_i}, \end{aligned}$$

which corresponds to a morphism $\Phi_\mathscr{A} \colon T_N \to \mathbb{C}^s$. Then $\operatorname{Spec}(\mathbb{C}[S])$ is an affine toric variety, which is isomorphic to $Y_\mathscr{A}$, and $\operatorname{Spec}(\mathbb{C}[S])$ is normal if and only if S is saturated[1].

2.1.2. *Examples.*

2.1.3. *Properties.*

Theorem 2.1.2. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex polyhedral cone. Then U_σ is smooth if and only if σ is smooth[2]. Moreover, all smooth affine toric varieties are of this form.

Proposition 2.1.2. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone and σ be a face of σ written as $\tau = H_m \cap \sigma$, where $m \in \sigma^\vee \cap M$. Then the semigroup algebra $\mathbb{C}[S_\tau]$ is the localization of $\mathbb{C}[S_\sigma]$ at $\chi^m \in \mathbb{C}[S_\sigma]$.

Proof. See Proposition 1.3.16 in [CLS11]. □

[1]See Definition 1.2.2.

[2]See Definition 1.3.11.