PARTIAL DIFFERENTIAL EQUATION

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Abstract.

Contents

0. Notations

1. U, V, W usually denotes open subsets in $\mathbb{R}^n$, where $n \geq 2$. We write $V \Subset U$ if $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact, and say V is compactly contained in U.
2. Function spaces
   (a) $C(\overline{U}) = \{u \in C(U) \mid u$ is uniformly continous on bounded subsets of $U\}$;
   (b) $C^k(\overline{U}) = \{u \in C^k(U) \mid D^\alpha u$ is uniformly continous on bounded subsets of U for all $|\alpha| \leq k\}$;
   (c) $C^\infty(\overline{U}) = \bigcap_{k=0}^{\infty} C^k(\overline{U})$.
3. $u(x) = u(x_1, \dots, x_n)$ is a smooth function defined on U, its partial differential $\frac{\partial u}{\partial x_i}$ is denoted by u_{x_i}.
4. Hessian of u is defined as a matrix $D^2 u(x) = (u_{x_i x_j})_{i \times j}$
5. Laplacian of u is defined as $\Delta u(x) = \operatorname{div} Du = \sum_{i=1}^n u_{x_i x_i}$
6. A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_i is a non-negative integer, is called a multiindex of order

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

7. Given a multiindex α, define

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

Part 1. Sobolev space

1. Sobolev space

1.1. Definitions and basic properties. Let $u, v \in L^1_{loc}(U)$ and α a multiindex. If

$$\int_U u D^\alpha \phi \mathrm{d}x = (-1)^{|\alpha|} \int_U v \phi \mathrm{d}x$$

for all test functions $\phi \in C_c^\infty(U)$, then we say v is the α-th weak partial derivatives of u, and denoted by $D^\alpha u = v$.

Remark 1.1.1. Note that u has a α-th weak partial derivative don't implies lower order weak partial derivatives. For example:

Lemma 1.1.1. Let $u \in L^1_{loc}(U)$. If a weak α-th partial derivative of u exists, then it's uniquely defined up to a set of measure zero.

Proof. Assume $v, \widetilde{v} \in L^1_{loc}(U)$ such that

$$\int_U u D^\alpha \phi \mathrm{d}x = (-1)^\alpha \int_U v \phi \mathrm{d}x = (-1)^\alpha \int_U \widetilde{v} \phi \mathrm{d}x$$

for all $\phi \in C_c^\infty(U)$, then

$$\int_U (v - \widetilde{v}) \phi \mathrm{d}x = 0$$

for all $\phi \in C_c^\infty(U)$. Thus from $v = \widetilde{v}$ a.e. □

Definition 1.1.1 (Sobolev space)**.**

$$W^{k,p}(U) := \{u \in L^p(U) \mid \forall |\alpha| \leq k, D^\alpha u \text{ exists and } D^\alpha u \in L^p(U)\}$$

Remark 1.1.2. Here we identify functions in $W^{k,p}(U)$ up to a set of measure zero.

Notation 1.1.1. If $p = 2$, we always write $H^k(U) = W^{k,2}(U)$ for $k \geq 0$. The letter H is used for $H^k(U)$ is a Hilbert space as we will see. Note that $H^0(U) = L^2(U)$.

Lemma 1.1.2. Assume $u, v \in W^{k,p}(U), |\alpha| \leq k$, then

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ for all multiindex α, β with $|\alpha| + |\beta| \leq k$.
2. For each $\lambda, \mu \in \mathbb{R}, \lambda u + \nu v \in W^{k,p}$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v, |\alpha| \leq k$.
3. If V is an open subset of U, then $u|_V \in W^{k,p}(V)$.
4. If $\xi \in C_c^\infty(U)$, then $\xi u \in W^{k,p}(U)$ and

$$D^\alpha(\xi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha - \beta} u$$

Proof. Let's check (4) in the case of $|\alpha| = 1$ as follows: for any $\phi \in C_c^\infty(U)$, we have

$$\begin{aligned}\int_U \xi u D^\alpha \phi &= \int_U u D^\alpha(\xi\phi) - u(D^\alpha \xi)\phi \mathrm{d}x \\ &= -\int_U (\xi D^\alpha u) + u D^\alpha \xi)\phi \mathrm{d}x\end{aligned}$$

which implies

$$D^\alpha(\xi u) = \xi D^\alpha u + u D^\alpha \xi$$

□

Definition 1.1.2 (norms)**.** For $u \in W^{k,p}(U)$, its norm is defined to be

$$\|u\|_{W^{k,p}(U)} := \begin{cases} (\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(U)}^p)^{\frac{1}{p}}, & 1 \le p < \infty \\ \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(U)}, & p = \infty \end{cases}$$

Remark 1.1.3. There is another equivalent norm defined as follows: For $u \in W^{k,p}(U)$,

$$\|u\|'_{W^{k,p}(U)} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(U)}$$

It's clear

$$\|u\|_{W^{k,p}(U)} = (\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(U)}^p)^{\frac{1}{p}} \le \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(U)} = \|u\|'_{W^{k,p}(U)}$$

Conversely,

$$\|u\|'_{W^{k,p}(U)} \le \sum_{|\alpha|\le k} \|u\|_{W^{k,p}(U)} = C(n,k)\|u\|_{W^{k,p}(U)}$$

where $C(n,k)$ is a constant depending on n, k.

Definition 1.1.3. Let $\{u_m\}, u \in W^{k,p}(U)$, we say

1. $u_m \to u$ in $W^{k,p}(U)$, if

$$\lim_{m\to\infty} \|u - u_m\|_{W^{k,p}(U)} = 0$$

2. $u_m \to u$ in $W^{k,p}_{loc}(U)$, if $u_m \to u$ in $W^{k,p}(V)$ for every $V \Subset U$.

Remark 1.1.4. It's clear that $u_m \to u$ in $W^{k,p}(U)$ if and only if $u_m \to u$ in $L^p(U)$ and for any $|\alpha| \le k$ we have $D^\alpha u_m \to D^\alpha u$ in $L^p(U)$.

Definition 1.1.4. We denote by $W_0^{k,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Theorem 1.1.1. For each $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.

Proof. It suffices to check $W^{k,p}(U)$ is complete. Assume $\{u_m\}$ is a Cauchy sequence in $W^{k,p}(U)$, then for each $|\alpha| \le k$, $\{D^\alpha u_m\}$ is a Cauchy sequence in $L^p(U)$, thus there exists $u_\alpha \in L^p(U)$ such that $D^\alpha u_m \to u_\alpha$ in $L^p(U)$ since $L^p(U)$ is complete. In particular, we have $u_m \to u$.

So it suffices to check $u \in W^{k,p}(U)$ with $D^\alpha u = u_\alpha$ for any $|\alpha| \leq k$, since we already have desired convergence in $L^p(U)$. Indeed, for any test function $\phi \in C_c^\infty(U)$, we have

$$\begin{aligned}
\int_U u D^\alpha \phi \mathrm{d}x &= \lim_{m\to\infty} \int u_m D^\alpha \phi \mathrm{d}x \\
&= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi \mathrm{d}x \\
&= (-1)^{|\alpha|} \int_U u_\alpha \phi \mathrm{d}x
\end{aligned}$$

This completes the proof. □

1.2. **Approximation.**

1.2.1. *Interior approximation by smooth functions.* Fix $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$.

Theorem 1.2.1 (local approximation by smooth functions)**.** Assume $u \in W^{k,p}(U)$ for some $1 \le p < \infty$, and set

$$u^\varepsilon(x) := \phi_\varepsilon * u(x), \quad x \in U_\varepsilon$$

then $u^\varepsilon \to u$ in $W^{k,p}_{loc}(U)$ as $\varepsilon \to 0$.

Proof. We have already senn in appendix C.5, $u^\varepsilon \to u$ in $L^p_{loc}(U)$. In order to show $u^\varepsilon \to u$ in $W^{k,p}_{loc}(U)$, it suffices to show $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p_{loc}(U)$ for any $|\alpha| \le k$. The following observation is crucial: For any $|\alpha| \le k$, then

$$D^\alpha u^\varepsilon = \phi_\varepsilon * D^\alpha u, \quad \text{in } U_\varepsilon$$

Indeed, for $x \in U_\varepsilon$

$$\begin{aligned} D^\alpha u^\varepsilon &= D^\alpha \int_U \phi_\varepsilon(x-y)u(y)\mathrm{d}y \\ &= \int_U D^\alpha_x \phi_\varepsilon(x-y)u(y)\mathrm{d}y \\ &= (-1)^{|\alpha|} \int_U D^\alpha_y \phi_\varepsilon(x-y)u(y)\mathrm{d}y \\ &= (-1)^{2|\alpha|} \int_U \phi_\varepsilon(x-y) D^\alpha_y u(y)\mathrm{d}y \\ &= (D^\alpha u)^\varepsilon \end{aligned}$$

From this observation we have $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p_{loc}(U)$. This completes the proof. $\square$

In fact, we can find smooth functions which approximate in $W^{k,p}(U)$ and not just in $W^{k,p}_{loc}(U)$.

Theorem 1.2.2 (global approximation)**.** Assume U is bounded and suppose $u \in W^{k,p}(U)$ for some $1 \le p < \infty$. Then there exists functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.

Proof. Let $U_i = \{x \in U \mid |x| < i, \operatorname{dist}(x, \partial U) > 1/i\}$, then $U = \bigcup_{i=1}^\infty U_i$. Write $V_i = U_{i+3} - \overline{U}_{i+1}$ and choose an open subset $V_0 \Subset U$ such that $U = \bigcup_{i=0}^\infty V_i$. Note that $\{V_i\}$ is a locally finite cover, thus we can choose a smooth partition of unity $\{\xi_i\}$ subordinate to it. Then for any $u \in W^{k,p}(U)$, we have $\xi_i u \in W^{k,p}(U)$ and $\operatorname{supp}(\xi_i u) \subset V_i$.

Fix $\delta > 0$, choose $\varepsilon_i > 0$ sufficiently small such that $u^i := \eta_{\varepsilon_i} * (\xi_i u)$ satisfies

$$\begin{cases} \|u^i - \xi_i u\|_{W^{k,p}(U)} \le \frac{\delta}{2^{i+1}}, & i \ge 0 \\ \operatorname{supp} u^i \subset W_i := U_{i+4} - \overline{U}_i, & i \ge 1 \end{cases}$$

Write $v := \sum_{i=0}^{\infty} u^i$, it's a well defined smooth function, since $\{W_i\}$ is locally finite. For each $V \Subset U$, there is $N \in \mathbb{N}$ such that $v = \sum_{i=0}^{N} u^i$, $u = \sum_{i=0}^{N} \xi_i u$, therefore

$$\begin{aligned}
\|v - u\|_{W^{k,p}(V)} &= \| \sum_{i=0}^{N} (u^i - \xi_i u) \|_{W^{k,p}(U)} \\
&\leq \sum_{i=0}^{N} \|u^i - \xi_i u\|_{W^{k,p}(U)} \\
&\leq \delta \sum_{i=0}^{N} \frac{1}{2^{i+1}} \\
&< \delta
\end{aligned}$$

□

1.2.2. *Global approximation up to the boundary.*

Theorem 1.2.3 (global approximation up to the boundary)**.** Let U be bounded and ∂U is C^1. Assume $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exists functions $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.

2. Extension theorem and trace theorem

2.1. Extension theorem.

Theorem 2.1.1 (extension theorem)**.** Suppose $1 \leq p \leq \infty$. Assume U is bounded and ∂U is C^k. Select a bounded open set V such that $U \Subset V$, then ther exists a bounded linear operator

$$E : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^n)$$

such that for each $u \in W^{k,p}(U)$, we have

1. $Eu = u$ almost everywhere in U;
2. $\operatorname{supp} Eu \subset V$;
3. $\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C(p, U, V)\|u\|_{W^{k,p}(U)}$, where $C(p, U, V)$ is a constant depending only on p, U and V.

2.2. Trace theorem.

Theorem 2.2.1. Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that

1. $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$;
2. $\|Tu\|_{L^p(\partial U)} \leq C(p, U)\|u\|_{W^{1,p}(U)}$ for each $u \in W^{1,p}(U)$ with constant $C(p, U)$ depending only on p and U.

Proof. **Step one:** Let's deal with flat boundary for $u \in C^1$. Assume $u \in C^1(\overline{U})$ and suppose $x \in \partial U$ and ∂U is flat near x, lying in the plane $\{x_n = 0\}$. Choose an open ball $B = (x, r)$ such that

$$\begin{cases} B^+ := B \cap \{x_n \geq 0\} \subset \overline{U} \\ B^- := B \cap \{x_n \leq 0\} \subset \mathbb{R}^n - U \end{cases}$$

Let $\widehat{B} = B(x, r/2), \Gamma = \partial U \cap \widehat{B}$. Select $\xi \in C_c^\infty(B)$ such that

1. $0 \leq \xi \leq 1$;
2. $\xi|_{\widehat{B}} = 1$

Let $x' = (x_1, \dots, x_{n-1}) \in \{x_n = 0\}$, then

$$\begin{aligned}
\int_\Gamma |u|^p \mathrm{d}x' &\leq \int_{x_n=0} \xi |u|^p \mathrm{d}x' \\
&= \int_{x_n=0} \int_0^\infty (-\xi |u|^p)_{x_n} \mathrm{d}x_n \mathrm{d}x' \\
&= -\int_{B^+} (-\xi |u|^p)_{x_n} \mathrm{d}x \\
&= -\int_{B^+} |u|^p \xi_{x_n} + p|u|^{p-1} \operatorname{sgn}(u) u_{x_n} \xi \mathrm{d}x \\
&\leq \|\xi_{x_n}\|_{L^\infty(B)} \int_{B^+} |u|^p \mathrm{d}x + p\|\xi\|_{L^\infty(B)} (|u|^p)^{(p-1)/p} |u_{x_n}| \mathrm{d}x \\
&\leq \|\xi_{x_n}\|_{L^\infty(B)} \int_{B^+} |u|^p \mathrm{d}x + p\|\xi\|_{L^\infty(B)} \int_B \frac{|u|^p}{p/(p-1)} + \frac{|u_{x_n}|^p}{p} \mathrm{d}x \\
&\leq C_1 \int_{B^+} |u|^p + |Du|^p \mathrm{d}x
\end{aligned}$$

where C_1 is a constant depending only on p and U, since bump function ξ depends on U.

Step two: If $x \in \partial U$ but ∂U is not flat near x, we straighten out the boundary near x to obtain the setting in step one, applying above estimate and changing variables, we obtain

$$\int_\Gamma |u|^p \mathrm{d}S \leq C_2 \int_U |u|^p + |Du|^p \mathrm{d}x$$

where Γ is some open subset of ∂U containing x, and C_2 is a constant depending only on p and U.

Step three: Since ∂U is compact, one can choose finitely many $x_i \in \partial U$ and open subset $\Gamma_i \subset \partial U$ such that $\partial U = \bigcup_{i=1}^N \Gamma_i$ and

$$\|u\|_{L^p(\Gamma_i)} \leq C_3 \|u\|_{W^{1,p}(U)}$$

where C_3 is a constant depending only on p and U. Consequently if we set $Tu := u|_{\partial u}$, then

$$\|Tu\|_{L^p(\partial U)} \leq C_3 \|u\|_{W^{1,p}(U)}$$

Step four: Assume $u \in W^{1,p}(U)$, then there exists a sequence $\{u_m\} \subset C^\infty(\overline{U})$ converging to u in $W^{1,p}(U)$. Since

$$\|Tu_m - Tu_l\|_{L^p(\partial U)} \leq C_3 \|u_m - u_l\|_{W^{1,p}(U)}$$

which implies that $\{Tu_m\}$ is a Cauchy sequence in $L^p(\partial U)$ and so we define

$$Tu := \lim_{m\to\infty} Tu_m$$

This limit is independent of the choice of $\{u_m\}$.

Step five: Now suppose $u \in C(\overline{U}) \cap W^{1,p}(U)$, note that $u_m \in C^\infty(\overline{U})$ converges uniformly to u on $\overline{U}$, thus $Tu = u|_{\partial U}$. $\square$

Theorem 2.2.2. Assume U is bounded and ∂U is C^1. Suppose $u \in W^{1,p}(U)$, then $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$ on ∂U.

Remark 2.2.1. This theorem characterizes the difference between $W^{1,p}(U)$ and $W_0^{1,p}(U)$.

3. Sobolev inequalities

Our goal in this section is to find embeddings of various Sobolev spaces into others. The crucial analytic tools here will be so-called "Sobolev type inequalities", which we will prove for smooth functions. These will then establish the estimates for arbitrary functions in various Sobolev spaces, since we already know smooth functions are dense.

To be explicit, if a function $u \in W^{1,p}(U)$, does u automatically belong to certain other spaces? The answer is yes, but in which spaces depends upon whether

$$W^{1,p}(U) \subset \begin{cases} L^{p^*}(U), & 1 \leq p < n \\ L^\infty(U), & p = n \\ C^{0,\gamma}(\overline{U}), & n < p \leq \infty \end{cases}$$

where p^* and γ are defined later.

3.1. Case $1 \leq p < n$.

Definition 3.1.1 (Sobolev conjugate)**.** For $1 \leq p < n$, we define its Sobolev conjugate as

$$p^* = \frac{np}{n-p}$$

Theorem 3.1.1 (Gagliardo-Nirenberg-Sobolev inequality)**.** Assume $1 \leq p < n$. Then

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p)\|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_c^1(\mathbb{R}^n)$, where $C(n,p)$ is a constant depending only on n and p.

Proof. Firstly we assume $p = 1$, then $p^* = n/(n-1)$. Since u is compactly supported, then for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_1} \partial_{x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \mathrm{d}y_i$$

So we have

$$|u(x)| \leq \int_{\mathbb{R}} |\partial_{x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \mathrm{d}y_i$$

Therefore

$$|u(x)|^{n/(n-1)} \leq \prod_{i=1}^n (\int_{\mathbb{R}} |\partial_{x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \mathrm{d}y_i)^{1/(n-1)}$$

Fix $x_2, \ldots, x_n$ and integrate above inequality with respect to x_1, we obtain

$$\int_{\mathbb{R}} |u|^{n/(n-1)} \mathrm{d}x_1 \leq \int_{\mathbb{R}} \prod_{i=1}^n (\int_{\mathbb{R}} |\partial_{x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \mathrm{d}y_i)^{1/(n-1)} \mathrm{d}x_1$$

Note that if we already fix $x_2, \ldots, x_n$, the following term is independent of x_1,

$$(\int_{\mathbb{R}} |\partial_{x_1} u(y_1, x_2, \ldots, x_n)| \mathrm{d}y_1)^{1/(n-1)}$$

Thus we have

$$\begin{aligned}\int_{\mathbb{R}} |u|^{n/(n-1)} \mathrm{d}x_1 &\leq \int_{\mathbb{R}} \prod_{i=1}^{n} (\int_{\mathbb{R}} |\partial_{x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \mathrm{d}y_i)^{1/(n-1)} \mathrm{d}x_1 \\ &= (\int_{\mathbb{R}} |\partial_{x_1} u| \mathrm{d}y_1)^{1/(n-1)} \int_{\mathbb{R}} \prod_{i=2}^{n} (\int_{\mathbb{R}} |\partial_{x_i} u| \mathrm{d}y_i)^{1/(n-1)} \mathrm{d}x_1\end{aligned}$$

Then by Hölder inequality we have

$$\int_{\mathbb{R}} \prod_{i=2}^{n} (\int_{\mathbb{R}} |\partial_{x_i} u| \mathrm{d}y_i)^{1/(n-1)} \mathrm{d}x_1 \leq (\prod_{i=2}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_{x_i} u| \mathrm{d}x_1 \mathrm{d}y_i)^{1/(n-1)}$$

All in all, we have

$$\int_{\mathbb{R}} |u|^{n/(n-1)} \mathrm{d}x_1 \leq (\int_{\mathbb{R}} |\partial_{x_1} u| \mathrm{d}y_1)^{1/(n-1)} (\prod_{i=2}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_{x_i} u| \mathrm{d}x_1)^{1/(n-1)}$$

Now fix $x_3, \ldots, x_n$ and integrate with respect to x_2, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \int_{\mathbb{R}} |u|^{n/(n-1)} \mathrm{d}x_1 \mathrm{d}x_2 \\ \leq &\underbrace{(\int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_{x_2} u| \mathrm{d}y_2 \mathrm{d}x_1)^{1/(n-1)}}_{\text{part I}} \int_{\mathbb{R}^n} (\int_{\mathbb{R}} |\partial_{x_1} u| \mathrm{d}y_1)^{\frac{1}{n-1}} \prod_{i=3}^{n} (\int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_{x_i} u| \mathrm{d}y_i \mathrm{d}x_1)^{\frac{1}{n-1}} \mathrm{d}x_2 \\ \leq &(\int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_{x_2} u| \mathrm{d}y_2 \mathrm{d}x_1)^{1/(n-1)} (\int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_{x_1} u| \mathrm{d}y_1 \mathrm{d}x_2)^{\frac{1}{n-1}}\end{aligned}$$

We can take part I outside of integration since it's independent of x_2 and for the second inequality we also use Hölder inequality.

Repeat this process with respect to $x_3, \ldots, x_n$, we will obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \mathrm{d}x &\leq \prod_{i=1}^{n} (\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| \mathrm{d}x_1 \ldots \mathrm{d}y_i \ldots \mathrm{d}x_n)^{\frac{1}{n-1}} \\ &= (\int_{\mathbb{R}^n} |Du| \mathrm{d}x)^{\frac{n}{n-1}}\end{aligned}$$

This shows desired estimate for $p = 1$.

Now consider $1 < p < n$, we apply above estimate to $v := |u|^{\gamma}$, where $\gamma > 1$ is to be selected. Then

$$\begin{aligned}(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n} |D|u|^{\gamma}| dx \\ &= \gamma \int_{\mathbb{R}^n} |u|^{\gamma - 1} |Du| dx \\ &\leq \gamma (\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx)^{\frac{p-1}{p}} (\int_{\mathbb{R}^n} |Du|^p dx)^{\frac{1}{p}}\end{aligned}$$

We choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$, in which case we have

$$\frac{\gamma n}{n-1} = \frac{np}{n-p} = p^*$$

This completes the proof. $\square$

Theorem 3.1.2 (estimates for $W^{1,p}, 1 \leq p < n$)**.** Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$, then for each $1 \leq q \leq p^*$, we have $u \in L^q(U)$ with estimate

$$\|u\|_{L^q(U)} \leq C(n,p,q,U)\|u\|_{W^{1,p}(U)}$$

where $C(n,p,q,U)$ is a constant depending only on n, p, q and U.

Proof. Let's firstly consider the case $q = p^*$. Since ∂U is C^1, then by extension theorem we have $Eu = \overline{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \overline{u} = u \text{ in } U \\ \operatorname{supp} \overline{u} \text{ is compact} \\ \|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C_1(p,U)\|u\|_{W^{1,p}(U)} \end{cases}$$

Because $\overline{u}$ has compact support, then there exists functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \to \overline{u} \quad \text{in } W^{1,p}(\mathbb{R}^n)$$

Now by Gagliardo-Nirenberg-Sobolev inequality we have

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C_2(n,p)\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$$

for all $m, l \geq 1$. Thus

$$u_m \to \overline{u} \quad \text{in } L^{p^*}(\mathbb{R}^n)$$

as well. Taking limit in Gagliardo-Nirenberg-Sobolev inequality $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C_2(n,p)\|Du_m\|_{L^p(\mathbb{R}^n)}$ we have

$$\|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C_2(n,p)\|D\overline{u}\|_{L^p(\mathbb{R}^n)}$$

Thus

$$\|u\|_{L^{p^*}(U)} \leq \|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C_2(n,p)\|D\overline{u}\|_{L^p(\mathbb{R}^n)} \leq C_1(p,U)C_2(n,p)\|Du\|_{L^p(U)}$$

Now for $1 \leq q \leq p^*$, one has interpolating inequality as follows

$$\begin{aligned} \|u\|^q_{L^q(U)} &= \int_U |u|^q \mathrm{d}x \\ &\leq (\int_U 1\mathrm{d}x)^{1-\frac{q}{p^*}}(\int_U (|u|^q)^{\frac{p^*}{q}}\mathrm{d}x)^{\frac{q}{p^*}} \\ &= |U|^{1-\frac{q}{p^*}}\|u\|^q_{L^{p^*}(U)} \end{aligned}$$

which implies $\|u\|_{L^q(U)} \leq C_3(p,q,U)\|u\|_{L^{p^*}(U)}$. Thus

$$\begin{aligned}\|u\|_{L^q(U)} &\leq C_3(p,q,U)\|u\|_{L^{p^*}(U)}\\ &\leq C_3(p,q,U)C_2(n,p)\|D\overline{u}\|_{L^p(\mathbb{R}^n)}\\ &\leq C_3(p,q,U)C_2(n,p)C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)}\\ &\leq C_3(p,q,U)C_2(n,p)CC_1(p,U)\|Du\|_{W^{1,p}(U)}\end{aligned}$$

This completes the proof. □

Remark 3.1.1. Here we need U is bounded with C^1 boundary to use extension theorem.

Theorem 3.1.3 (estimates for $W_0^{1,p}, 1 \leq p < n$)**.** Let U be a bounded, open subset of $\mathbb{R}^n$. Assume $1 \leq p < n$ and $u \in W_0^{1,p}(U)$, then for each $1 \leq q \leq p^*$, we have the following estimates

$$\|u\|_{L^p(U)} \leq C(n,p,q,U)\|Du\|_{L^p(U)}$$

where C is a constant depending only on n, p, q and U.

Proof. Since $u \in W_0^{1,p}(U)$, there exist functions $u_m \in C_c^\infty(U)$ converging to u in $W^{1,p}(U)$. We extend each function u_m to be zero on $\mathbb{R}^n \backslash \overline{U}$ and mimick above proof to obtain

$$\|u\|_{L^{p^*}(U)} \leq C_1(n,p)\|Du\|_{L^p(U)}$$

As U is bounded, by interpolating inequality one has

$$\|u\|_{L^q(U)} \leq C_2(q,U)\|u\|_{L^{p^*}(U)}$$

This completes the proof. □

Remark 3.1.2. Since $p \leq p^*$, so in particular, for all $1 \leq p < n$, we have

$$\|u\|_{L^p(U)} \leq C(n,p,U)\|Du\|_{L^p(U)}$$

This estimate is sometimes called Poincaré inequality.

3.2. **Case** $p = n$**.** Owing to our estimate for $1 \leq p < n$, you know that if $u \in W^{1,p}(U)$, then $u \in L^{p^*}(U)$ where $p^* = \frac{np}{n-p}$. Since $p^* \to \infty$ as $p \to n$, you might expect $u \in L^\infty(U)$, if $u \in W^{1,n}(U)$. However, this is false when $n > 1$. For example, if $U = B(0,1)$, the function $u = \log\log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(U)$ but not to $L^\infty(U)$.

3.3. **Case** $n < p \leq \infty$**.** In this case, we will show if $u \in W^{1,p}(U)$, then u is in fact Hölder continous, after possibly being redefined on a set of measure zero.

Theorem 3.3.1 (Morrey's inequality)**.** Assume $n < p \leq \infty$, then there exists a constant C, depending only on p and n such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R})} \leq C\|u\|_{W^{1,p}(\mathbb{R})}$$

for all $u \in C^1(\mathbb{R})$, where $\gamma = 1 - n/p$.

Proof. We claim that there exists a constant C_1, depending only on n such that

$$\fint_{B(x,r)} |u(y)-u(x)|\mathrm{d}y \leq C_1 \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\mathrm{d}y$$

for any ball $B(x,r) \subset \mathbb{R}^n$.

Fix $w \in \partial B(0,1)$, then if $0 < s < r$, we have

$$\begin{aligned} |u(x+sw)-u(x)| &= |\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} u(x+tw)\mathrm{d}t| \\ &= |\int_0^s Du(x+tw)w\mathrm{d}t| \\ &\leq \int_0^s |Du(x+tw)|\mathrm{d}t \end{aligned}$$

Hence

$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|\mathrm{d}S(w)$$

Now choose any $x, y \in \mathbb{R}^n$ and write $r := |x-y|, W := B(x,r) \cap B(y,r)$. Then we have

$$\begin{aligned} \fint_W |u(x)-u(z)|\mathrm{d}z &\leq \frac{\mu(B(x,r))}{\mu(W)} \fint_{B(x,r)} |u(x)-u(z)|\mathrm{d}z \\ &\leq C_5 (\int_{B(x,r)} |Du|^p \mathrm{d}z)^{\frac{1}{p}} (\int_{B(x,r)} \frac{\mathrm{d}z}{|x-z|^{(n-1)p/(p-1)}})^{(p-1)/p} \\ &\leq C_5 (\frac{1}{n-(n-1)p/(p-1)} r^{n-(n-1)p/(p-1)})^{(p-1)/p} \|Du\|_{L^p(\mathbb{R}^n)} \\ &= C_6 r^{(p-n)/p} \|Du\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

Similarly you have

$$\fint_W |u(y)-u(z)|\mathrm{d}z \leq C_6 r^{(p-n)/p} \|Du\|_{L^p(\mathbb{R}^n)}$$

Thus

$$\begin{aligned} |u(x)-u(y)| &\leq \fint_W |u(x)-u(z)|\mathrm{d}z + \fint_W |u(y)-u(z)|\mathrm{d}z \\ &\leq C_6 r^{(p-n)/p} \|Du\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

Therefore

$$\begin{aligned} [u]_{C^{0,1-n/p}(\mathbb{R}^n)} &= \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \\ &\leq C_7 \|Du\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

□

Definition 3.3.1. u^* is called a version of a given function u, if $u = u^*$ a.e.

Theorem 3.3.2 (estimates for $W^{1,p}, n < p \leq \infty$)**.** Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1 - \frac{n}{p}$ with estimate

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|u\|_{W^{1,p}(U)}$$

The constant C depends only on p, n and U.

Proof. Since ∂U is C^1, then by extension theorem we have $Eu = \overline{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \overline{u} = u \text{ in } U \\ \operatorname{supp} \overline{u} \text{ is compact} \\ \|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)} \end{cases}$$

Now assume $n < p < \infty$, since $\overline{u}$ has compact support, then there exists functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \to \overline{u} \quad \text{in } W^{1,p}(\mathbb{R}^n)$$

Now by Morrey's inequality we have

$$\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C_1\|u_m - u_l\}_{W^{1,p}(\mathbb{R}^n)}$$

for all $m, l \geq 1$, where $\gamma = 1 - n/p$. Thus there exists $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ such that

$$u_m \to u^* \quad \text{in } C^{0,\gamma}(\mathbb{R}^n)$$

Since $u_m \to \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$, we have $u^* = u$ a.e. in U. Taking limit in Morrey's inequality we have

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C_1\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)}$$

Thus

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq \|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C_1\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$$

This completes the proof for $n < p < \infty$. For case $p = \infty$, it's easy to check directly. $\square$

Remark 3.3.1. The proof here is almost the same as the proof in estimate for $1 \leq p < n$.

3.4. **General Sobolev inequalities.**

Theorem 3.4.1 (general Sobolev inequalities)**.** Let U be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Assume $u \in W^{k,p}$, then

1. If $kp < n$, then $u \in L^q(U)$, where $1/q = 1/p - k/n$. Furthermore, we have estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}$$

   where C depends on k, p, n and U.

2. If $kp > n$, then $u \in C^{k-[\frac{n}{p}-1],\gamma}(\overline{U})$, where

$$\gamma = \begin{cases} [\frac{n}{p} + 1 - \frac{n}{p}], & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer} \end{cases}$$

Furthermore, we have estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}}(\overline{U}) \leq C\|u\|_{W^{k,p}(U)}$$

where C depends on k, p, n, γ and U.

3.5. **Compact embedding.** Let X, Y be Banach spaces, $f : X \to Y$ is an injective bounded linear operator.

Definition 3.5.1 (compact operator)**.** f is a compact operator, if each bounded sequence in X is precompact in Y, that is any bounded sequence $\{u_m\} \subset X$ has a subsequnce whose image under f converges in Y to some limit v.

Definition 3.5.2 (compact embedding)**.** X is compactly embedded in Y via f, if f is a compact operator. If inclusion $i : X \to Y$ is a bounded compact operator, we write $X \Subset Y$.

Theorem 3.5.1 (Rellich-Kondrachov compactness theorem)**.** Assume U is a bounded open subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $1 \leq p < n$. Then

$$W^{1,p}(U) \Subset L^q(U)$$

for each $1 \leq q < p^*$.

Proof. Fix $1 \leq q < p^*$ and note that since U is bounded, then Gagliardo-Nirenberg-Sobolev inequality implies

$$W^{1,p}(U) \subset L^{p^*}(U) \subset L^q(U), \quad \|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}$$

So it remains to show that any bounded sequence $\{u_m\} \subset W^{1,p}(U)$, there exists a subsequnce which converges in $L^q(U)$.

By extension theorem we may assume $U = \mathbb{R}^n$ and $\{u_m\}$ all have support in some bounded open set $V \subset \mathbb{R}^n$, and

$$\sup_m \|u_m\|_{W^{1,p}(V)} \leq M < \infty \tag{3.1}$$

Furthermore, we may assume $\{u_m^\varepsilon\}$ all have support in V as well.

Step one: A crucial observation is the following convergence is uniformly in m:

$$u_m^\varepsilon \to u_m, \quad \text{in } L^q(V)$$

To prove this, we first assume that u_m is smooth, then

$$\begin{aligned} u_m^\varepsilon - u_m(x) &= \int_{B(0,1)} \eta(y)(u_m(x-\varepsilon y) - u_m(x))\mathrm{d}y \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(u_m(x-\varepsilon ty))\mathrm{d}t\mathrm{d}y \\ &= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\varepsilon ty)y\mathrm{d}t\mathrm{d}y \end{aligned}$$

thus

$$\begin{aligned} \int_V |u_m^\varepsilon(x) - u_m(x)|\mathrm{d}x &\leq \varepsilon \int_V \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x-\varepsilon ty)|\mathrm{d}t\mathrm{d}y\mathrm{d}x \\ &= \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x-\varepsilon ty)|\mathrm{d}x\mathrm{d}t\mathrm{d}y \\ &\leq \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(z)|\mathrm{d}z\mathrm{d}t\mathrm{d}y \\ &= \varepsilon \int_V |Du_m(z)|\mathrm{d}z \end{aligned}$$

By approximation this estimate holds if $u_m \in W^{1,p}(V)$. Hence

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \leq \varepsilon \|Du_m\|_{L^1(V)} \leq \varepsilon C_1 \|Du_m\|_{L^p(V)}$$

where the latter inequality holding since V is bounded. Furthermore, $\|Du_m\|_{L^p(V)}$ is bounded uniformly in m by (3.1). Thus we have $u_m^\varepsilon \to u_m$ in $L^1(V)$ uniformly in m.

But since $1 \leq q < p^*$, we see using the interpolation inequality that

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta}$$

By Gagliardo-Nirenberg-Sobolev inequality we have

$$\|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta} \leq 2\|u_m\|_{L^{p^*}(V)}^{1-\theta} \leq 2C\|u_m\|_{W^{1,p}(V)}^{1-\theta}$$

Together with (3.1) we can see

$$\|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta} \leq 2C_2 M^{1-\theta}$$

thus bounded uniformly in m. So

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \varepsilon C_1 C_2 M^{2-\theta}$$

which completes the proof of our claim.

Remark 3.5.1. Note that $\|u_m^\varepsilon - u_m\|_{L^{p^*}(V)} \to 0$ as $\varepsilon \to 0$ may not be uniformly in m, that's why we can just prove compact embedding for $q < p^*$. For $q = p^*$, there is an example such that $W^{1,p}(U) \subset L^{q^*}(U)$ is not compact.

Step two: we claim that: For each fixed $\varepsilon > 0$, the sequence $\{u_m^\varepsilon\}$ is uniformly bounded and equicontinous. Indeed, if $x \in \mathbb{R}^n$, then

$$\begin{aligned}|u_m^\varepsilon(x)| &\leq \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u_m(y)|\mathrm{d}y \\ &\leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \\ &\leq \frac{C}{\varepsilon^n} < \infty\end{aligned}$$

for arbitrary m. Similarly

$$\begin{aligned}|Du_m^\varepsilon(x)| &\leq \int_{B(x,\varepsilon)} |D\eta_\varepsilon(x-y)||u_m(y)|\mathrm{d}y \\ &\leq |D\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \\ &\leq \frac{C}{\varepsilon^{n+1}} < \infty\end{aligned}$$

for arbitrary m.

Step three: Now fix $\delta > 0$, we claim there exists a subsequnce $\{u_{m_j}\} \subset \{u_m\}$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta$$

Indeed, we first use assertion in step one to select $\varepsilon > 0$ sufficiently small such that

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \frac{\delta}{2}$$

for arbitrary m. However, we have already shown that $\{u_m^\varepsilon\}$ satisfies the condition for Arzela-Ascoli compactness criterion in step two, thus we can obtain a subsequnce $\{u_{m_j}^\varepsilon\} \subset \{u_m^\varepsilon\}$ which converges uniformly on V. In particular, we have

$$\limsup_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} = 0$$

Thus we have

$$\begin{aligned}\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq& \limsup_{j,k\to\infty} \|u_{m_j} - u_{m_j}^\varepsilon\|_{L^q(V)} + \limsup_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} \\ &+ \limsup_{j,k\to\infty} \|u_{m_k}^\varepsilon - u_{m_k}\|_{L^q(V)} \\ \leq& \delta\end{aligned}$$

Step four: Now use assertion in step three with $\delta = 1, \frac{1}{2}, \frac{1}{3}$ and use a standard diagonal argument to extract a subsequnce $\{u_{m_l}\} \subset \{u_m\}$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0$$

This completes the proof. $\square$

Remark 3.5.2. Now let's consider $p \geq n$ under asumption of this theorem.

1. For $p = n$, one has $W^{1,n}(U) \Subset L^q(U)$ for any $1 \leq q < \infty$. Indeed, if $f \in W^{1,n}(U)$, then $f \in W^{1,p}(U)$ for all $p < n$. Thus for each such p, apply the result for $p < n$ we have
$$W^{1,n}(U) \Subset L^q(U)$$
for any $1 \leq q < np/(n-p)$, since bounded composed with compact is compact. However, $np/(n-p)$ can be arbitrary large by taking $p \to n$, thus we obtain the desired result.
2. For $n < p < \infty$, one has
$$W^{1,p}(U) \subset W^{1,n}(U) \Subset L^q(U)$$
for $1 \leq q < \infty$

In particular, we can see
$$W^{1,p}(U) \Subset L^p(U)$$
for $1 \leq p < \infty$.

4. Additional topics

4.1. **Poincaré inequality.**

Notation 4.1.1. $(u)_U = \fint_U u\mathrm{d}y$ is used to denote the average of u over U.

Lemma 4.1.1. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \quad \text{a.e. in } U$$

Then u is constant a.e. in U.

Proof. Consider $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$. For $x \in U_\varepsilon$, consider $u^\varepsilon = \int_{U_\varepsilon} \eta_\varepsilon(x-y)u(y)\mathrm{d}y$, then u^ε is smooth and

$$Du_\varepsilon = \int_{U_\varepsilon} \eta_\varepsilon(x-y)Du(y)\mathrm{d}y$$

Since $Du = 0$ a.e., we have that $Du_\varepsilon = 0$ for all $x \in U_\varepsilon$ and hence u^ε is constant in U_ε. Since $\|u^\varepsilon \to u\|_{L^p(U)} \to 0$ as $\varepsilon \to 0$, we have that u is constant a.e. in U. □

Theorem 4.1.1 (Poincaré inequality)**.** Let U be a bounded, connected open subset of $\mathbb{R}^n$ with C^1 boundary. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending on n, p and U, such that

$$\|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

for each $u \in W^{1,p}(U)$.

Proof. We argue by contradiction. Were the stated estimate false, then for each positive integer $k \in \mathbb{N}_{\geq 0}$, there exists a function $u_k \in W^{k,p}(U)$ satisfying

$$\|u_k - (u_k)_U\|_{L^p(U)} > k\|Du\|_{L^p(U)}$$

We renormalize by defining

$$v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}$$

Then we have

$$(v_k)_U = 0, \|v_k\|_{L^p(U)} = 1$$

and by choice of v_k we have

$$\|Dv_k\|_{L^p(U)} < \frac{1}{k}$$

In particular, $\{v_k\}$ are bounded in $W^{1,p}(U)$. In view of proof of Rellich-Kondrachov theorem, there exists a subsequnce $\{v_{k_j}\}$ of $\{v_k\}$ and a function $v \in L^p(U)$ such that

$$v_{k_j} \to v \quad \text{in } L^p(U)$$

Thus we have

$$(v)_U = 0, \|v\|_{L^p(U)} = 1$$

On the other hand, $\|Dv_k\|_{L^p(U)} < \frac{1}{k}$ implies for rach $i = 1, 2, \dots, n$ and $\phi \in C_c^\infty(U)$, we have

$$\int_U v\phi_{x_i} = \lim_{j\to\infty}\int_U v_{k_j}\phi_{x_i}\mathrm{d}x = -\lim_{j\to\infty}\int_U v_{k_j,x_i}\phi\mathrm{d}x = 0$$

Consequently $v \in W^{1,p}(U)$ with $Dv = 0$ a.e. Thus by above lemma we have v is constant since U is connected. However, this is a contradiction: since v is constant and $(v)_U = 0$ implies $v \equiv 0$, which contradicts to $\|v\|_{L^p(U)} = 1$. □

4.2. **Difference quotients.** Assume $u : U \to \mathbb{R}$ is a locally summable function and $V \Subset U$.

Definition 4.2.1. The i-th difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + e_i h) - u(x)}{h}$$

for $x \in V$ and $h \in \mathbb{R}, 0 < |h| < \mathrm{dist}(V, \partial U)$.

Notation 4.2.1. $D^h u := (D_1^h u, \dots, D_n^h u)$.

Lemma 4.2.1 (integration by parts)**.** For $i = 1, \dots, n$ and $\phi \in C_c^\infty(V)$, we have

$$\int_V u(D_i^h\phi)\mathrm{d}x = -\int_V (D_i^{-h}u)\phi\mathrm{d}x$$

Proof. Note that for small enough h we have

$$\int_V u(x)\phi(x + he_i)\mathrm{d}x = \int_{V+he_i} u(x - he_i)\phi(x)\mathrm{d}x$$

and since ϕ has compact support in V, so the latter integral effectively only extends over a subset of V, so we have

$$\begin{aligned}\int_V u(x)[\frac{\phi(x+he_i) - \phi(x)}{h}]\mathrm{d}x &= \frac{1}{h}(\int_V u(x)\phi(x+he_i)\mathrm{d}x - \int_V u(x)\phi(x)\mathrm{d}x)\\ &= \frac{1}{h}(\int_V u(x-he_i)\phi(x)\mathrm{d}x - \int_V u(x)\phi(x)\mathrm{d}x)\\ &= -\int_V [\frac{u(x) - u(x-he_i)}{h}]\phi(x)\mathrm{d}x\end{aligned}$$

This completes the proof. □

Theorem 4.2.1. Suppose $1 \le p < \infty$ and $u \in W^{1,p}(U)$. Then for each $V \Subset U$,

$$\|D^h u\|_{L^p(V)} \le C\|Du\|_{L^p(U)}$$

for constant C and all $0 < |h| < \frac{1}{2}\,\mathrm{dist}(V, \partial U)$.

Theorem 4.2.2. Assume $1 < p < \infty$ and $u \in L^p(V)$, there exists a constant C such that

$$\|D^h u\|_{L^p(V)} \le C$$

for all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$. Then

$$u \in W^{1,p}(V)$$

with $\|Du\|_{L^p(V)} \leq C$.

Proof. Estimates for $\|D^h u\|_{L^p(V)}$ implies

$$\sup_h \|D_i^{-h} u\|_{L^p(V)} < \infty$$

for all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$. Thus there exists a function $v_i \in L^p(V)$ and a subsequnce $h_k \to 0$ such that

$$D_i^{-h_k} uharpoonupv_i$$

in $L^p(V)$, since $L^p(V)$ is reflexive when $1 < p < \infty$. So we have

$$\begin{aligned}
\int_V u\phi_{x_i} &= \int_U u\phi_{x_i} \mathrm{d}x \\
&= \lim_{h_k \to 0} \int_U u D_i^{h_k} \phi \mathrm{d}x \\
&= -\lim_{h_k \to 0} \int_V D_i^{-h_k} u\phi \mathrm{d}x \\
&= -\int_V v_i \phi \mathrm{d}x \\
&= -\int_U v_i \phi \mathrm{d}x
\end{aligned}$$

This implies $Du \in L^p(V)$, we deduce $u \in W^{1,p}(V)$ as $u \in L^p(V)$. □

4.3. **Other spaces of functions.**

4.3.1. *The space H^{-1}.*

Definition 4.3.1. The dual space of $H_0^1(U)$ is denoted by $H^{-1}(U)$.

Remark 4.3.1. We have the following inclusions

$$H_0^1(U) \subset L^2(U) \subset H^{-1}(U)$$

Notation 4.3.1. We use $\langle \text{-}, \text{-} \rangle$ to denote the pairing between $H^{-1}(U)$ and $H_0^1(U)$.

Definition 4.3.2. For $f \in H^{-1}(U)$, the norm of f is defined as

$$\|f\|_{H^{-1}(U)} := \sup\{\langle f, u\rangle \mid u \in H_0^1(U), \|u\|_{H_0^1(U)} \leq 1\}$$

Theorem 4.3.1. Here are some characterizations of H^{-1}.
1. Assume $f \in H^{-1}(U)$, then there exists $f^0, \ldots, f^n \in L^2(U)$ such that

$$\langle f, v\rangle = \int f^0 v + \sum_{i=1}^n f^i v_{x_i} \mathrm{d}x$$

for $v \in H_0^1(U)$. Furthermore, one has

$$\|f\|_{H^{-1}(U)} = \inf(\int_U \sum_{i=0}^n |f^i|^2)^{\frac{1}{2}}$$

2. $(u,v)_{L^2(U)} = \langle u, v\rangle$ for all $u \in H_0^1(U), v \in L^2(U) \subset H^{-1}(U)$.

4.3.2. *Spaces involving time.* Let X be a real Banach space, with norm $\| \ \|$.

Definition 4.3.3. The space $L^p([0,T];X)$ consists of all strongly measurable functions $\mathbf{u} : [0,T] \to X$ such that

$$\|\mathbf{u}\|_{L^p([0,T];X)} := (\int_0^T \|\mathbf{u}(t)\|^p \mathrm{d}t)^{\frac{1}{p}} < \infty$$

for $1 \le p < \infty$ and

$$\|\mathbf{u}\|_{L^\infty([0,T];X)} := \mathrm{esssup}_{0\le t\le T} \|\mathbf{u}(t)\| < \infty$$

Definition 4.3.4. The spaceee $C([0,T];X)$ consists of continous functions $\mathbf{u} : [0,T] \to X$ such that

$$\|\mathbf{u}\|_{C([0,T];X)} := \max_{0\le t\le T} \|\mathbf{u}(t)\| < \infty$$

Definition 4.3.5. Let $\mathbf{u} \in L^1([0,T];X)$, $\mathbf{v} \in L^1([0,T];X)$ is called the weak derivative of $\mathbf{u}$, written $\mathbf{u}' = \mathbf{v}$, if

$$\int_0^T \phi'(t)\mathbf{u}(t)\mathrm{d}t = -\int_0^T \phi(t)\mathbf{v}(t)\mathrm{d}t$$

for all scalar test functions $\phi \in C_c^\infty([0,T])$.

Definition 4.3.6. The Sobolev space $W^{1,p}([0,T];X)$ consists of all functions $\mathbf{u} \in L^p([0,T];X)$ such that $\mathbf{u}' \in L^p([0,T];X)$. Furthermore,

$$\|\mathbf{u}\|_{W^{1,p}([0,T];X)} := \begin{cases} (\int_0^T \|\mathbf{u}(t)\|^p + \|\mathbf{u}'(t)\|^p \mathrm{d}t)^{\frac{1}{p}}, & 1 \le p < \infty \\ \mathrm{esssup}_{0\le t\le T}(\|\mathbf{u}(t)\| + \|\mathbf{u}'(t)\|), & q = \infty \end{cases}$$

Theorem 4.3.2. Suppose $\mathbf{u} \in L^2([0,T];H_0^1(U))$ with $\mathbf{u}' \in L^2([0,T];H^{-1}(U))$, then

$$u \in C([0,T];L^2(U))$$

Part 2. Second-order elliptic equations

5. Introductions

In this part, we always assume U is an open, bounded subset of $\mathbb{R}^n$.

5.1. What is elliptic equation?

Let L denote a second-order partial differential operator having either the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_j})_{x_i} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{5.1}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{5.2}$$

for given coefficients functions a^{ij}, b^i, c defined on U.

Definition 5.1.1. L is called

1. in divergence form, if L is given in (5.1);
2. in non-divergence form, if L is given in (5.2).

Remark 5.1.1. If the highest-order coefficients a^{ij} are C^1 functions, then an operator given in divergence form can be rewritten into non-divergence form, and vice versa. The operator in different forms has its own advantages:

1. The divergence form is most natural for energy method, based on integration by parts;
2. The non-divergence form is most appropriate for maximum priciples.

Furthermore, in this part we always assume our differential operator L has the following algebraic property.[1]

Definition 5.1.2 (uniformly ellipticity)**.** A partial differential operator L is uniformly elliptic if

1. a^{ij} is symmetric;
2. There exists a constant $\theta > 0$ such that
$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$
for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

Remark 5.1.2. In other words, uniformly ellipticity means $(a^{ij}(x))_{ij}$ is a symmetric matrix, and $(a^{ij}(x))_{ij}$ is positive definite, with smallest eigenvalue greater than or equal θ for a.e. $x \in U$.

[1] As always, the heart of each computation is the invocation of ellipticity: the point is to derive analytic estimates from the structural, algebraic assumption of ellipticity.

5.2. **How to solve it?** Firstly let's consider Dirichlet problem in divergence form, that is a PDE given by

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{5.3}$$

where

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_j})_{x_i} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

In Sobolev spaces, it's much easier to find "solutions" of (5.3) in some weak sense. The solution of this kind is sometimes called weak solution or generalized solution. To explain its motivation, assume we already have a smooth solution of (5.3), then multiply this equation by a smooth test function $v \in C_c^\infty(U)$ and integrate by parts, we have

$$\int_U \sum_{i,j=1}^{n} (a^{ij}u_{x_j})v_{x_i} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv\mathrm{d}x = \int_U fv\mathrm{d}x$$

By approximation we can show the same identity holds with the smooth function v replaced by any $v \in H_0^1(U)$, and the resulting identity makes sense if only $u \in H_0^1(U)$.

If we associated the Dirichlet problem (5.3) the bilinear form

$$B[u,v] := \int_U \sum_{i,j=1}^{n} (a^{ij}u_{x_j})v_{x_i} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv\mathrm{d}x$$

for all $u, v \in H_0^1(U)$. Then the above argument motivate us to define weak solution as follows.

Definition 5.2.1 (weak solution)**.** For $f \in H^{-1}(U)$, $u \in H_0^1(U)$ is a weak solution of Dirichlet problem (5.3) if

$$B[u,v] = \langle f, v \rangle$$

for all $v \in H_0^1(U)$, where $\langle \text{-}, \text{-} \rangle$ denotes the pairing of $H_0^1(U)$ with its dual space.

We follows the following three steps to back to classical solutions:

1. Existence and uniqueness of a weak solution is established by Lax-Milgram theorem;
2. The weak solution is proved to be smooth under appropriate assumptions. This is a regularity result;
3. A classical solution is recovered by showing that any smooth weak solution is a classical solution.

6. The existence and uniqueness of weak solution

6.1. **Lax-Milgram theorem.** Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\text{-},\text{-})$, and $\langle\text{-},\text{-}\rangle$ denotes the pairing of H with its dual space.

Definition 6.1.1 (bounded)**.** A bilinear functional $B: H\times H\to\mathbb{R}$ is called bounded, if

$$|B[u,v]|\leq \alpha\|u\|\|v\|$$

for all $u,v\in H$, where α is a finite constant.

Definition 6.1.2 (coercive)**.** A bilinear functional $B: H\times H\to\mathbb{R}$ is called coercive, if

$$|B[u,u]|\geq \beta\|u\|^2$$

for all $u\in H$, where β is a constant >0.

Theorem 6.1.1 (Lax-Milgram theorem)**.** Assume $B: H\times H\to\mathbb{R}$ is a bounded and coercive bilinear functional and $f: H\to\mathbb{R}$ is a bounded linear functional on H, then there exists a unique $v\in H$ such that

$$B[u,v]=\langle f,v\rangle$$

for all $v\in H$.

6.2. **Energy estimates: A baby version.** In this section, we try to verify the hypothesis of the Lax-Milgram theorem for

$$B[u,v]:=\int_U\sum_{i,j=1}^n(a^{ij}u_{x_j})v_{x_i}+\sum_{i=1}^n b^iu_{x_i}v+cuv\mathrm{d}x$$

for all $u,v\in H^1_0(U)$ under assumptions:
1. $a^{ij},b^i,c\in L^\infty(U)$;
2. $f\in L^2(U)$.

Theorem 6.2.1. Under the assumptions in this section, there exist constants $C(n,L),\overline{\mu}(n,L)>0$ depending only on n and operator L such that
1. $|B[u,v]|\leq C(n,L)\|u\|_{H^1_0(U)}\|v\|_{H^1_0(U)}$
2. $B[u,u]\geq\frac{\theta}{2}\|Du\|^2_{L^2(U)}-\overline{\mu}(n,L)\|u\|^2_{L^2(U)}$

for all $u,v\in H^1_0(U)$.

Proof. For (1). It's easy to see

$$\begin{aligned}|B[u,v]|&\leq\sum_{i,j=1}^n\|a^{ij}\|_{L^\infty(U)}\int_U|Du||Dv|\mathrm{d}x+\sum_{i=1}^n\|b^i\|_{L^\infty(U)}\int_U|Du||v|\mathrm{d}x+\|c\|_{L^\infty(U)}\int_U|u||v|\mathrm{d}x\\&\leq C(n,L)\|u\|_{H^1_0(U)}\|v\|_{H^1_0(U)}\end{aligned}$$

For (2). The uniformly ellipticity implies

$$\begin{aligned}\theta\int_U |Du|^2\mathrm{d}x &\le \int_U \sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j}\mathrm{d}x\\ &= B[u,u]-\int_U\sum_{i=1}^n b^iu_{x_i}u+cu^2\mathrm{d}x\\ &\le B[u,u]+\sum_{i=1}^n\|b^i\|_{L^\infty(U)}\int_U|Du||u|\mathrm{d}x+\|c\|_{L^\infty(U)}\int_U u^2\mathrm{d}x\end{aligned}$$

Now from Cauchy's inequality one has

$$\int_U |Du||u|\mathrm{d}x\le\varepsilon\int_U|Du|^2\mathrm{d}x+\frac{1}{4\varepsilon}\int_U u^2\mathrm{d}x$$

where ε is a constant >0. Choose ε sufficiently small such that

$$\varepsilon\sum_{i=1}^n\|b^i\|_{L^\infty(U)}<\frac{\theta}{2}$$

Thus

$$\frac{\theta}{2}\int_U|Du|^2\mathrm{d}x\le B[u,u]+(\|c\|_{L^\infty(U)}+\frac{1}{4\varepsilon})\int_U u^2\mathrm{d}x$$

that is

$$B[u,u]\ge\frac{\theta}{2}\|Du\|^2_{L^2(U)}-\overline{\mu}(n,L)\|u\|^2_{L^2(U)}$$

since ε depends only on n and L. □

Remark 6.2.1. By using Poincaré inequality, one can deduce the following estimates

$$\beta\|u\|^2_{H^1_0(U)}\le B[u,u]+\gamma\|u\|^2_{L^2(U)}$$

for appropriate constants $\beta>0$ and $\gamma\ge0$.

Corollary 6.2.1. There exists $\overline{\mu}>0$ such that for all $\mu>\overline{\mu}$, there exists a unique weak solution $u\in H^1_0(U)$ of the boundary-value problem

$$\begin{cases}Lu+\mu u=f & \text{in } U\\ u=0 & \text{on } \partial U\end{cases}$$

where $f\in H^{-1}(U)$.

Proof. Take $\overline{\mu}$ as $\overline{\mu}(n,L)$ in Theorem 6.2.1 and define the following bilinear form

$$B_\mu[u,v]:=B[u,v]+\mu(u,v),\quad u,v\in H^1_0(U)$$

which corresponds to the operator $L_\mu:=Lu+\mu u$. Then B_μ[-,-] satisfies the hypothesis of the Lax-Milgram theorem. □

6.3. **Energy estimates: A general version.** In above section, we put quite strong requirements on the coefficients of operator L. In fact, we can consider the following assumptions:

1. U is an open bounded subset of $\mathbb{R}^n, n \geq 3$ with C^1 boundary[2].
2.

$$\sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} + \sum_{i=1}^n \|b^i\|_{L^n(U)} + \|c\|_{L^{\frac{n}{2}}(U)} \leq \Lambda \tag{6.1}$$

Theorem 6.3.1. Under the assumptions in this section, there exist constant $C(n, \Lambda, U), \overline{\mu}(n, \theta, U) > 0$ such that

$$\begin{aligned} |B[u,v]| &\leq C(n,\Lambda,U)\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)} \\ |B[u,u]| &\geq \frac{\theta}{2}\|Du\|_{L^2(U)}^2 - \overline{\mu}(n,\theta,U)\|u\|_{L^2(U)}^2 \end{aligned}$$

for all $u, v \in H_0^1(U)$.

Proof. For (1). By generalized Hölder inequality and Sobolev inequality

$$\|u\|_{L^{2^*}(U)} \leq C_1(n,U)\|u\|_{H^1(U)}$$

where $2^* = \frac{2n}{n-2}$. Holding these tools, we have the following estimates

1.
$$|\int_U \sum_{i,j=1}^n a^{ij}(x) u_{x_j} v_{x_i} \mathrm{d}x| \leq C_2(n,\Lambda)\|Du\|_{L^2(U)}\|Dv\|_{L^2(U)}$$

2.
$$\begin{aligned} |\int_U \sum_{i=1}^n b^i(x) u_{x_i} v(x) dx \,| &\leq \sum_{i=1}^n \|b^i\|_{L^n(U)}\|Du\|_{L^{2^*}(U)}\|v\|_{L^2(U)} \\ &\leq C_3(n,\Lambda,U)\|Du\|_{L^2(U)}\|v\|_{H_0^1(U)} \end{aligned}$$

3.
$$\begin{aligned} |\int_U c(x) uv \mathrm{d}x| &\leq \|c\|_{L^{\frac{n}{2}}(U)}\|u\|_{L^{2^*}(U)}\|v\|_{L^{2^*}(U)} \\ &\leq C_4(n,\Lambda,U)\|u\|_{H^1(U)}\|v\|_{H_0^1(U)} \end{aligned}$$

This completes the estimates for upper bound of $|B[u,v]|$.

For (2). Recall for $f \in L^p(U), 1 \leq p < \infty$, there exists f_1, f_2 such that $f = f_1 + f_2$ and

$$\|f_1\|_{L^p(U)} \leq \varepsilon, \quad \|f_2\|_{L^\infty(U)} \leq K(\varepsilon)$$

So there exists decomposition

$$b^j = b_1^j + b_2^j, \quad c = c_1 + c_2$$

[2]We need U is open bounded with C^1 boundary to use Sobolev embedding, and $n \geq 3$ is just a technique condition.

such that

$$\sum_{j=1}^{n}(\|b_1^i\|_{L^n(U)} + \|c_1\|_{L^{\frac{n}{2}}(U)}) < \varepsilon$$

$$\sum_{j=1}^{n}(\|b_1^i\|_{L^\infty(U)} + \|c_1\|_{L^\infty(U)}) < K(\varepsilon)$$

By the same estimate in (1) it's easy to see

$$\begin{aligned}B_1[u,u] &:= \int_U \sum_{i,j=1}^{n}(a^{ij}u_{x_j})u_{x_i} + \sum_{i=1}^{n} b_1^i u_{x_i} u + c_1 u^2 \mathrm{d}x \\ &\geq \theta\|Du\|^2_{L^2(U)} - \varepsilon C(n,U)\|u\|^2_{H_0^1(U)}\end{aligned}$$

And by Young's inequality

$$\begin{aligned}B_2[u,u] &:= \int_U \sum_{i=1}^{n} b_2^i u_{x_i} u + c_2 u^2 \mathrm{d}x \\ &\geq -C(n)K(\varepsilon)\int_U |Du||u| + u^2 \mathrm{d}x \\ &\geq -\frac{\theta}{4}\|Du\|^2_{L^2(U)} - C(n)K(\varepsilon)(\frac{C(n)K(\varepsilon)}{\theta} + 1)\|u\|^2_{L^2(U)}\end{aligned}$$

Choose ε such that $\varepsilon C(n,U) = \frac{\theta}{4}$ and set

$$\overline{\mu} = \frac{\theta}{4} + C(n)K(\varepsilon)(\frac{C(n)K(\varepsilon)}{\theta} + 1)$$

then we have

$$B[u,u] = B_1[u,u] + B_2[u,u] \geq \frac{\theta}{2}\|Du\|^2_{L^2(U)} - \overline{\mu}\|u\|^2_{L^2(U)}$$

□

6.4. **Fredholm alternative.** Now we're going to use Fredholm theory for compact operator to glean more detailed information regarding the solvablity of second-order elliptic PDE. The Fredholm alternative theorem in a Hilbert space is stated as follows:

Theorem 6.4.1 (Fredholm alternative theorem)**.** Let H be a Hilbert space, $K : H \to H$ is a compact operator with adjoint operator K^*, then

1. Precisely one of the following statements holds:
   (a) For all $f \in H$, $u - Ku = f$ has unique solution in H;
   (b) $u - Ku = 0$ has non-zero solution in H.
2. $\dim\ker(I - K) = \dim\ker(I - K^*)$, where I is identity operator;
3. For all $f \in H$, $u - Ku = f$ has solution in H if and only if $f \in \ker(I - K^*)^{\perp}$.

Now we are going to consider Dirichlet problem (5.3) by Fredholm alternative theorem. The process is divided into three parts.

Step one: The adjoint operator of L. Recall the adjoint operator of L, denoted by L^* is defined as

$$\langle Lu, v\rangle = \langle u, L^*v\rangle$$

where $u, v \in H_0^1(U)$. To be explicit, we have

$$\begin{aligned}\langle Lu, v\rangle &= \int_U \sum_{i,j=1}^n (a^{ij}u_{x_j})v_{x_i} + \sum_{i=1}^n b^i u_{x_i} v + cuv\mathrm{d}x \\ &\overset{(1)}{=} \int_U \sum_{i,j=1}^n (a^{ij}v_{x_j})u_{x_i} - \sum_{i=1}^n b^i v_{x_i} u + (c - \sum_{i=1}^n b^i_{x_i})vu\mathrm{d}x\end{aligned}$$

where (1) holds from integration by parts. Thus adjoint operator L^* can be written explicitly as

$$L^*v = -\sum_{i,j=1}^n (a^{ij}v_{x_j})_{x_i} - \sum_{i=1}^n b^i v_{x_i} + (c - \sum_{i=1}^n b^i_{x_i})v$$

Step two: The equivalent expression of $Lu = f$. Let $H = H^{-1}(U)$ and $L_\mu u = Lu + \mu u$, by Theorem 6.3.1 we can choose $\mu > \overline{\mu}$ such that for all $f \in H$, $L_\mu u = f$ has unique solution in $H_0^1(U)$, that is there exists

$$L_\mu^{-1} : H \to H_0^1 \subset H$$

So the following statements are equivalent:

1. $Lu = f$ has unique solution in $H_0^1(U)$;
2. $L_\mu u = f + \mu u$ has unique solution in $H_0^1(U)$;
3. $u = L_\mu^{-1}(f + \mu u)$ has unique solution in $H_0^1(U)$;
4. $u - Ku = h$ has unique solution in $H_0^1(U)$, where

$$K = \mu L_\mu^{-1}, \quad h = L_\mu^{-1} f = \frac{1}{\mu} K f \in H_0^1$$

Step three: K is a compact operator.

Step four: Conclusion. Apply Fredholm alternative to $H = H^{-1}(U)$ and K defined in step two.

1. Precisely one of the following statements holds:
   (a) For all $u \in H^{-1}(U)$, $u - Ku = h$ has unique solution in H, which is equivalent to for all $f \in H^{-1}(U)$, $Lu = f$ has unique solution in $H_0^1(U)$ by step two;
   (b) $u - Ku = 0$ has non-zero solution in H, which is equivalent to $Lu = 0$ has non-zero solution in $H_0^1(U)$.
2. $\dim\ker(I - K) = \dim\ker(I - K^*)$, where I is identity operator. By definition we have

$$\begin{aligned}\ker(I - K) &= \{u \in H_0^1(U) \mid Lu = 0\} \\ \ker(I - K^*) &= \{u \in H_0^1(U) \mid L^*u = 0\}\end{aligned}$$

3. For all $f \in H$, the following statements are equivalent:
   (a) $Lu = f$ has solution in $H_0^1(U)$;
   (b) $u - Ku = h$, where h is defined as step two, has solution in H;
   (c) $h \in \ker(I - K^*)^{\perp}$, this holds from Fredholm alternative;
   (d) $\langle h, v\rangle = 0$ for all $v \in \ker(I - K^*)$. Furthermore,
$$\langle h, v\rangle = \langle \frac{1}{\mu} Kf, v\rangle = \frac{1}{\mu}\langle f, K^* v\rangle = \frac{1}{\mu}\langle f, v\rangle = \frac{1}{\mu}\int_U fv \mathrm{d}x$$
   where the last equality holds only if $f \in L^2(U)$.

All in all, we have proven:

Theorem 6.4.2. Let U be an open bounded subset of $\mathbb{R}^n, n \geq 3$ with C^1 boundary, L is a uniformly elliptic operator with coefficients satisfying (6.1). Then

1. Precisely one of the following statements holds:
   (a) For all $f \in H^{-1}(U)$, $Lu = f$ has unique weak solution in $H_0^1(U)$;
   (b) $Lu = 0$ has non-zero weak solution in $H_0^1(U)$.
2. $\dim(\{u \in H_0^1(U) \mid Lu = 0\}) = \dim(\{u \in H_0^1(U) \mid L^* u = 0\}) < \infty$;
3. For all $f \in L^2(U)$, $Lu = f$ has unique weak solution in $H_0^1(U)$ if and only if
$$\int_U fv \mathrm{d}x = 0$$
   for all $v \in \{u \in H_0^1(U) \mid L^* u = 0\}$.

7. Regularity

In this section, we're going to deal with regularity of weak solutions of Dirichlet problem (5.3).

7.1. Interior regularity.

Theorem 7.1.1 (Interior H^2-regularity)**.** Assume

1. $a^{ij} \in C^1(U), b^i, c \in L^\infty(U)$;
2. $f \in L^2(U)$.

Suppose $u \in H^1(U)$ is a weak solution of

$$Lu = f \quad \text{in } U$$

Then $u \in H^2_{loc}(U)$ and for each $V \Subset U$ we have

$$\|u\|_{H^2(V)} \leq C(L, U, V)(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

where $C(L, U, V)$ is a constant depending only on U, V and coefficients of L.

Remark 7.1.1. Note here we do not require $u \in H^1_0(U)$, since we're doing interior estimate, and we don't require boundary-value.

Proof. Fix any open set $V \Subset U$ and choose an open set W such that $V \Subset W \Subset U$. Then select a smooth function ξ satisfying

$$\begin{cases} \xi \equiv 1 & \text{on } V \\ \xi \equiv 0 & \text{on } \mathbb{R}^n - W \\ 0 \leq \xi \leq 1 & \end{cases}$$

Such ξ is called a cutoff function. Its purpose in the subsequent calculations will be to restrict all expression to the subset W, which is a positive distance away from ∂U. This is necessary as we have no information concerning the behavior of u near ∂U.

Now assume u is a weak solution, that is we have $B[u, v] = (f, v)$ for all $v \in H^1_0(U)$. Consequently,

$$\underbrace{\sum_{i,j}^{n} \int_U a^{ij} u_{x_i} v_{x_j} \mathrm{d}x}_{\text{denoted by } A} = \underbrace{\int_U \widetilde{f} v \mathrm{d}x}_{\text{denoted by } B}$$

where

$$\widetilde{f} := f - \sum_{i=1}^{n} b^i u_{x_i} - cu$$

Now let's take $v := -D_k^{-h}(\xi^2 D_k^h u)$ to estimate A and B.

1. Estimate of A: We have

$$\begin{aligned}
A &= -\sum_{i,j=1}^{n}\int_U a^{ij}u_{x_i}[D_k^{-h}(\xi^2 D_k^h u)]_{x_j}\mathrm{d}x \\
&= \sum_{i,j=1}^{n}\int_U D_k^h(a^{ij}u_{x_i})(\xi^2 D_k^h u)_{x_j}\mathrm{d}x \\
&= \sum_{i,j=1}^{n}\int_U a^{ij,h}(D_k^h u_{x_i})(\xi^2 D_k^h u)_{x_j} + (D_k^h a^{ij})u_{x_i}(\xi^2 D_k^h u)_{x_j} \\
&= \underbrace{\sum_{i,j=1}^{n}\int_U a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j}\xi^2\mathrm{d}x}_{\text{denoted by } A_1} \\
&\quad + \underbrace{\sum_{i,j=1}^{n}\int_U \{a^{ij,h}D_k^h u_{x_i} D_h^k u 2\xi\xi_{x_j} + (D_k^h a^{ij}u_{x_i} D_k^h u_{x_j}\xi^2 + (D_k^h a^{ij})u_{x_i} D_k^h u 2\xi\xi_{x_j}\}\mathrm{d}x}_{\text{denoted by } A_2}
\end{aligned}$$

The uniformly elliptic condition implies

$$A_1 \geq \theta\int_U \xi^2|D_k^h Du|^2\mathrm{d}x$$

2.

Thus we have

$$\int_V |D_k^h Du|^2\mathrm{d}x \leq \int_U \xi^2|D_k^2 Du|^2\mathrm{d}x \leq C\int_U f^2 + u^2 + |Du|^2\mathrm{d}x$$

for $k = 1, 2, \ldots, n$ and all sufficiently small $|h| \neq 0$. Thus by Theorem we deduce $Du \in H^1_{loc}(U)$ and thus $u \in H^2_{loc}(U)$ with estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)})$$

□

Theorem 7.1.2 (Higher interior regularity)**.** For a non-negative integer m, we assume
1. $a^{ij}, b^i, c \in C^{m+1}(U)$;
2. $f \in H^m(U)$.

Suppose $u \in H^1(U)$ is a weak solution of $Lu = f$ in U, then $u \in H^{m+2}_{loc}(U)$ and for each $V \Subset U$ we have

$$\|u\|_{H^{m+2}(V)} \leq C(m, L, U, V)(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$$

where $C(m, L, U, V)$ is a constant depending only on m, U, V and coefficients of L.

Theorem 7.1.3 (Infinite differentiability in the interior)**.** Assume
1. $a^{ij}, b^i, c \in C^\infty(U)$;

2. $f \in C^\infty(U)$.
Suppose $u \in H^1(U)$ is a weak solution of $Lu = f$ in U, then $u \in C^\infty(U)$.

Proof. Thanks to higher interior regularity, one has $u \in H^m_{loc}$ for each positive integer m. Hence by general Sobolev inequalities, one has $u \in C^k(U)$ for all $k \geq [\frac{n}{2}] + 1$. In particular, u is smooth. □

7.2. **Boundary regularity.**

Theorem 7.2.1 (Boundary H^2-regularity)**.** Assume
1. $a^{ij} \in C^1(\overline{U}), b^i, c \in L^\infty(U)$;
2. $f \in L^2(U)$;
3. ∂U is C^2.

Suppose $u \in H^1_0(U)$ is a weak solution of

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Then $u \in H^2(U)$ with estimate

$$\|u\|_{H^2(U)} \leq C(L, U)(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

where $C(L, U)$ is a constant depending only on U and coefficients of L.

Theorem 7.2.2 (Higher boundary regularity)**.** Assume
1. $a^{ij}, b^i, c \in C^{m+1}(\overline{U})$;
2. $f \in H^m(U)$;
3. ∂U is C^{m+2}.

Suppose $u \in H^1_0(U)$ is a weak solution of

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Then $u \in H^{m+2}(U)$ with estimate

$$\|u\|_{H^{m+2}(U)} \leq C(m, L, U)(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$$

where $C(m, L, U)$ is a constant depending only on m, U and coefficients of L.

Theorem 7.2.3 (Infinite differentiability in the interior)**.** Assume
1. $a^{ij}, b^i, c \in C^\infty(\overline{U})$;
2. $f \in C^\infty(\overline{U})$;
3. ∂U is C^∞.

Suppose $u \in H^1_0(U)$ is a weak solution of

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

then $u \in C^\infty(\overline{U})$.

8. Maximum priciples

In this section we consider the maximum priciple for second-order elliptic partial differential equations in non-divergence form, that is

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

where coefficients are continous.

Maximum priciple methods based on the observation that if a C^2 function u attains its maximum over an open set U at a point $x_0 \in U$, then

$$Du(x_0) = 0$$
$$D^2u(x_0) \leq 0$$

Deductions based on above facts are consequently called “pointwise” in literature, and thus utterly different from the integral-based energy estimate.

8.1. **Weak maximum priciple.**

Theorem 8.1.1 (weak maximum priciple)**.** Assume $u \in C^2(U) \cap C(\overline{U})$ and $c \equiv 0$ in U.

1. If $Lu \leq 0$ in U, then

$$\max_{\overline{U}} u = \max_{\partial U} u$$

2. If $Lu \geq 0$ in U, then

$$\min_{\overline{U}} u = \min_{\partial U} u$$

Remark 8.1.1. A function satisfies $Lu \leq 0$ is called a subsolution, and is called supersolution if $Lu \geq 0$.

Proof. It suffices to prove (1), since if $Lu \geq 0$, then $L(-u) \leq 0$, then apply (1) to conclude.

Let's first suppose we have the strict inequality

$$Lu < 0$$

in U and there exists a point $x_0 \in U$ with $u(x_0) = \max_{\overline{U}} u$. Now at this point we have

$$Du(x_0) = 0$$
$$D^2u(x_0) \leq 0$$

Since $A = (a^{ij}(x_0))$ is symmetric and positive definite, there exists an orthogonal matrix $O = (o_{ij})$ such that

$$OAO^T = \mathrm{diag}\{d_1, \ldots, d_n\}$$

where $d_k > 0$ for all $k = 1, \dots, n$. Write $y = x_0 + O(x - x_0)$, then $x - x_0 = O^T(y - x_0)$, so

$$\begin{aligned} u_{x_i} &= \sum_{k=1}^{n} u_{y_k} o_{ki} \\ u_{x_i x_j} &= \sum_{k,l=1}^{n} u_{y_k y_l} o_{ki} o_{lj} \end{aligned}$$

Hence at point x_0 one has

$$\begin{aligned} \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} a^{ij} u_{y_k y_l} o_{ki} o_{lj} \\ &= \sum_{k=1}^{n} d_k u_{y_k y_k} \\ &\leq 0, \end{aligned}$$

since $d_k > 0$ and $D^2 u(x_0) \leq 0$. Thus we have

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} \geq 0$$

holds at point x_0. A contradiction.

In general case, consider

$$u^\varepsilon(x) := u(x) + \varepsilon e^{\lambda x_1}$$

where $\lambda > 0$ is to be selected. Then

$$\begin{aligned} Lu^\varepsilon &= Lu + \varepsilon L(e^{\lambda x_1}) \\ &\leq \varepsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1) \\ &\leq \varepsilon e^{\lambda x_1}(-\lambda\theta \|b\|_{L^\infty(U)} \lambda) \\ &< 0 \end{aligned}$$

provided λ is sufficiently large. According to previous case one has

$$\max_{\overline{U}} u^\varepsilon = \max_{\partial U} u^\varepsilon$$

Then let $\varepsilon \to 0$ to conclude. $\square$

Theorem 8.1.2 (weak maximum priciple)**.** Assume $u \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ in U.

1. If $Lu \leq 0$ in U, then
$$\max_{\overline{U}} u \leq \max_{\partial U} u^+$$
2. If $Lu \geq 0$ in U, then
$$\min_{\overline{U}} u \geq -\max_{\partial U} u^-$$

8.2. **Strong maximum priciple.** Now let's strengthen the foregoing assertions, that is a subsolution u can't attain its maximum at an interior point of a connected region at all, unless u is constant. This is called strong maximum priciple, which depends on the following subtle lemma.

Lemma 8.2.1 (Hopf's lemma)**.** Assume $u \in C^2(U) \cap C^1(\overline{U})$ and $Lu \leq 0$ in U. If there exists a point $x_0 \in \partial U$ such that

$$u(x_0) > u(x)$$

for all $x \in U$, and there exists an open ball $B \subset U$ with $x_0 \in \partial B$. Then

1. If $c \equiv 0$, then
$$\frac{\partial u}{\partial \nu}(x_0) > 0$$
where ν is the outer unit normal to B at x_0.
2. If $c \geq 0$, the same conclusion holds if $u(x_0) \geq 0$.

Theorem 8.2.1 (strong maximum priciple)**.** Assume $u \in C^2(U) \cap C(\overline{U})$ and $c \equiv 0$ in U, and we assume U is connected.

1. If $Lu \leq 0$ in U, and u attains its maximum over $\overline{U}$ at an interior point, then u is constant in U;
2. If $Lu \geq 0$ in U, and u attains its minimum over $\overline{U}$ at an interior point, then u is constant in U.

8.3. **Harnack's inequality.** Harnack's inequality states that the values of a non-negative solution are comparable, at least in any subregion away from the boundary.

Theorem 8.3.1. Assume $u \geq 0$ is a C^2 solution of $Lu = 0$ in U, and suppose $V \Subset U$ is connected. Then there exists a constant $C(L, V)$ depending only on L, V such that

$$\sup_V u \leq C(L, V) \inf_V u$$

Part 3. Linear evolution equations

9. Second-order parabolic equations

Second-order parabolic equations are natural generalizations of the heat equation. In this section we will study the existence and uniqueness of appropriately defined weak solutions, their smoothness and other properties.

In this section we assume U to be open, bounded subset of $\mathbb{R}^n$ and $U_T = U \times (0, T]$ for some fixed time $T > 0$ with boundary Γ_T.

9.1. Definitions.

We will first study the following initial/boundary-value problem

$$\begin{cases} u_t + Lu = f \text{ in } U_T \\ \qquad u = 0 \text{ on } \partial U \times [0, T] \\ \qquad u = g \text{ on } U \times \{t = 0\}, \end{cases} \tag{9.1}$$

where $f : U_T \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are given and $u(x, t) : \overline{U}_T \to \mathbb{R}$ is the unknown. The letter L denotes for each time t a second-order partial differential operator, having either the divergence form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x, t) u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x, t) u_{x_i} + c(x, t) u$$

or else the non-divergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x, t) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x, t) u_{x_i} + c(x, t) u$$

for given coefficients a^{ij}, b^i, c. Similarly, we also assume L satisfies the following property.

Definition 9.1.1 (uniformly parabolic)**.** The partial differential operator $\partial_t + L$ is uniformly parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x, t) \xi_i \xi_j \geq \theta |\xi|^2$$

for all $(x, t) \in U_T, \xi \in \mathbb{R}^n$.

Example 9.1.1. If $a^{ij} = \delta_{ij}, b^i = c = f = 0$, in which case $L = -\Delta$, then $u_{tt} + Lu = 0$ becomes the heat equation.

9.2. Motivation and definition of weak solutions.

Mimicking the developments for elliptic equations, we assume L has the divergence form and we also assume

$$\begin{aligned} & a^{ij}, b^i, c \in L^\infty(U_T) \quad (i, j = 1, \ldots, n) \\ & f \in L^2(U_T) \\ & g \in L^2(U) \end{aligned}$$

for convenience. Firstly let's define the following time-dependent bilinear form

$$B[u, v; t] := \int_U \sum_{i,j=1}^n a^{ij}(\text{-}, t) u_{x_i} v_{x_j} + \sum_{i=1}^n b^i(\text{-}, t) u_{x_i} v + c(\text{-}, t) uv dx$$

for $u, v \in H_0^1(U)$ and $0 \le t \le T$.

Remark 9.2.1 (motivation for definition of weak solution). Suppose $u(x, t)$ is a smooth solution of (9.1) and defining the associated mapping

1. $\mathbf{u} : [0, T] \to H_0^1(U)$, given by $[\mathbf{u}(t)](x) := u(x, t)$;
2. $\mathbf{u}' : [0, T] \to H_0^1(U)$, given by $[\mathbf{u}'(t)](x) := u_t(x, t)$;
3. $\mathbf{f} : [0, T] \to L^2(U)$, given by $[\mathbf{f}(t)](x) := f(x, t)$.

Now fix any function $v \in H_0^1(U)$, multiply the equation $u_t + Lu = f$ by v and integrate by parts, one obtain

$$(\mathbf{u}', v) + B[\mathbf{u}, v; t] = (\mathbf{f}, v)$$

To be explicit, for each $0 \le t \le T$, one has

$$(u_t, v) = \int_U g^0 v + \sum_{i=1}^n g^i v_{x_i} \mathrm{d}x$$

where $g^0 := f - \sum_{i=1}^n b^i u_{x_i} - cu$ and $g^j := \sum_{i=1}^n a^{ij} u_{x_i}$. According to Theorem 4.3.1, for each $0 \le t \le T$, one has $u_t \in H^{-1}(U)$ with

$$\|u_t\|_{H^{-1}(U)} \le (\sum_{j=0}^n \|g^j\|_{L^2(U)}^2)^{\frac{1}{2}} \le C(\|u\|_{H_0^1(U)} + \|f\|_{L^2(U)}) < \infty$$

This motivate us to find weak solution with $\mathbf{u}' \in L^2([0, T]; H^{-1}(U))$.

Definition 9.2.1. A function $\mathbf{u} \in L^2([0, T]; H_0^1(U))$ with $\mathbf{u}' \in L^2([0, T]; H^{-1}(U))$ is called a weak solution of (9.1) if

1. $$\langle \mathbf{u}', v \rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v)$$
   for each $v \in H_0^1(U)$ and a.e. $0 \le t \le T$;
2. $$\mathbf{u}(0) = g$$

Remark 9.2.2. Thanks to Theorem 4.3.2, $\mathbf{u}(0) = g$ makes sense.

9.3. **Unique existence of weak solution.** We intend to build a weak solution of the parabolic problem (9.1) by first constructing solutions of certain finite dimensional approximations and then passing to limits. This is called Galerkin's method.

To be explicit, assume $w_k, k = 1, \ldots$ are smooth and $\{w_k\}_{k=1}^\infty$ is an orthogonal basis of $H_0^1(U)$ and $L^2(U)$.

Theorem 9.3.1. For each positive integer m, there exists a unique $\mathbf{u}_m$ of the form

$$\mathbf{u}_m(t) := \sum_{k=1}^m d_m^k(t) w_k$$

such that
1. $d_m^k(0) = (g, w_k), k = 1, \dots m$;
2. $(\mathbf{u}'_m, w_k) + B[\mathbf{u}_m, w_k; t] = (\mathbf{f}, w_k)$ for $0 \le t \le T$ and $k = 1, \dots, m$.

Proof. Note that

$$(\mathbf{u}'_m, w_k) = (d_m^k)'(t)$$
$$B[\mathbf{u}_m, w_k; t] = \sum_{l=1}^m e^{kl} d_m^l(t)$$

where $e^{kl} = B[w_l, w_k; t]$. Then we can write (2) as

$$(d_m^k)'(t) + \sum_{l=1}^m e^{kl}(t) d_m^l(t) = f^k(t)$$

where $f^k(t) = (\mathbf{f}(t), w_k)$. It's a linear system of ODE with initial conditions. According to standard existence theory of ODE, this completes the proof. □

We propose now to send m to infinity and to show a subsequnce of our solutions $\mathbf{u}_m$ converges to a weak solution of (9.1). For this we need some uniform estimate.

Theorem 9.3.2 (energy estimate)**.** There exists a constant $C(U, T, L)$ such that

$$\max_{0 \le t \le T} \|\mathbf{u}_m(t)\|_{L^2(U)} + \|\mathbf{u}_m\|_{L^2([0,T];H_0^1(U))} + \|\mathbf{u}'_m\|_{L^2([0,T];H^{-1}(U))} \le C(U, T, L)(\|\mathbf{f}\|_{L^2([0,T];L^2(U))} + \|g\|_{L^2(U)})$$

Proof. **Step one:** Note that we have $(\mathbf{u}'_m, w_k) + B[\mathbf{u}_m, w_k; t] = (\mathbf{f}, w_k)$ for $0 \le t \le T$ and $1 \le k \le m$, multiply this equation by $d_m^k(t)$ and take sumation for $k = 1, \dots, m$, one has

$$(\mathbf{u}'_m, \mathbf{u}_m) + B[\mathbf{u}_m, \mathbf{u}_m; t] = (\mathbf{f}, \mathbf{u}_m)$$

holds for $0 \le t \le T$. By Remark 6.2.1, there exists constant $\beta > 0, \gamma \ge 0$ such that

$$\beta \|\mathbf{u}_m\|^2_{H_0^1(U)} \le B[\mathbf{u}_m, \mathbf{u}_m; t] + \gamma \|\mathbf{u}_m\|^2_{L^2(U)}$$

holds for $0 \le t \le T$ and $m \ge 1$. Furthermore, note that
1. $|(\mathbf{f}, \mathbf{u}_m)| \le \frac{1}{2}\|\mathbf{f}\|^2_{L^2(U)} + \frac{1}{2}\|\mathbf{u}_m\|^2_{L^2(U)}$;
2. $(\mathbf{u}'_m, \mathbf{u}_m) = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\|\mathbf{u}_m\|^2_{L^2(U)})$.

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{u}_m\|^2_{L^2(U)}) + 2\beta \|\mathbf{u}_m\|^2_{H_0^1(U)} \le C_1 \|\mathbf{u}_m\|^2_{L^2(U)} + C_2 \|\mathbf{f}\|^2_{L^2(U)} \tag{9.2}$$

holds for $0 \le t \le T$ and appropriate constants C_1 and C_2.

Step two: Now set

$$\eta(t) := \|\mathbf{u}_m(t)\|^2_{L^2(U)}$$
$$\xi(t) := \|\mathbf{f}(t)\|^2_{L^2(U)}$$

Then (9.2) implies

$$\eta'(t) \leq C_1\eta(t) + C_2\xi(t)$$

holds for a.e. $0 \leq t \leq T$. Thus by Gronwall's inequality one has

$$\eta(t) \leq e^{C_1 t}(\eta(0) + C_2 \int_0^t \xi(s)\mathrm{d}s)$$

where $0 \leq t \leq T$. Note that $\eta(0) = \|\mathbf{u}_m(0)\|^2_{L^2(U)} \leq \|g\|^2_{L^2(U)}$, then

$$\max_{0\leq t\leq T} \|\mathbf{u}_m\|^2_{L^2(U)} \leq C(\|g\|_{L^2(U)})^2 + \|\mathbf{f}\|^2_{L^2([0,T];L^2(U))}$$

Step three: Integrate (9.2) from 0 to T and employ the inequality obtained in Step two, one has

$$\begin{aligned}\|\mathbf{u}_m\|^2_{L^2([0,T];H^1_0(U))} &= \int_0^T \|\mathbf{u}_m\|^2_{H^1_0(U)}\mathrm{d}t \\ &\leq C(\|g\|^2_{L^2(U)} + \|\mathbf{f}\|^2_{L^2([0,T];L^2(U))})\end{aligned}$$

Step four: Fix any $v \in H^1_0(U)$ with $\|v\|_{H^1_0(U)} \leq 1$, and write $v = v_1 + v_2$, where $v_1 \in \operatorname{span}\{w_k\}_{k=1}^m$ and $(v_2, w_k) = 0$ for all $1 \leq k \leq m$. Furthermore, $\|v_1\|_{H^1_0(U)} \leq \|v\|_{H^1_0(U)} \leq 1$, since $\{w_k\}_{k=0}^\infty$ are orthogonal in $H^1_0(U)$. □

Holding this estimate, we are going to pass to limits as $m \to \infty$ to build a weak solution of (9.1). Before that, we need the following lemma.

Lemma 9.3.1. Suppose

$$\begin{cases}\mathbf{u}_m \rightharpoonup \mathbf{u} & \text{in } L^2([0,T];H^1_0(U)) \\ \mathbf{u}'_m \rightharpoonup \mathbf{v} & \text{in } L^2([0,T];H^{-1}(U))\end{cases}$$

Then $\mathbf{v} = \mathbf{u}'$.

Theorem 9.3.3 (existence of weak solution)**.** There exists a weak solution of (9.1)

Proof. **Step one:** According to energy estimate, we find the sequence $\{\mathbf{u}_m\}_{m=1}^\infty$ is bounded in $L^2([0,T];H^1_0(U))$ and $\{\mathbf{u}'_m\}_{m=1}^\infty$ is bounded in $L^2([0,T];H^{-1}(U))$. Consequently there exists a subsequnce $\{\mathbf{u}_{m_l}\}_{l=1}^\infty$ and a function $\mathbf{u} \in L^2([0,T];H^1_0(U))$ with $\mathbf{u}' \in L^2([0,T];H^{-1}(U))$ such that

$$\begin{cases}\mathbf{u}_m \rightharpoonup \mathbf{u} & \text{in } L^2([0,T];H^1_0(U)) \\ \mathbf{u}'_m \rightharpoonup \mathbf{u}' & \text{in } L^2([0,T];H^{-1}(U))\end{cases}$$

Step two: Fix an integer N and choose a function $\mathbf{v} \in C^1([0,T]; H_0^1(U))$ having the form

$$\mathbf{v}(t) = \sum_{k=1}^{N} d^k(t) w_k \tag{9.3}$$

where $d_k(t)$ are given smooth functions. Choose $m \geq N$ and multiply the following equality

$$(\mathbf{u}_m', w_k) + B[\mathbf{u}_m, w_k; t] = (\mathbf{f}, w_k)$$

by $d^k(t)$, sum $k = 1, \dots, N$, and then integrate with respect to t to get

$$\int_0^T \langle \mathbf{u}_m', \mathbf{v}\rangle + B[\mathbf{u}_m, \mathbf{v}; t]\mathrm{d}t = \int_0^T (\mathbf{f}, \mathbf{v})\mathrm{d}t \tag{9.4}$$

Take $m = m_l$ and pass to weak limits we have

$$\int_0^T \langle \mathbf{u}', \mathbf{v}\rangle + B[\mathbf{u}, \mathbf{v}; t]\mathrm{d}t = \int_0^T (\mathbf{f}, \mathbf{v})\mathrm{d}t \tag{9.5}$$

This equality holds for all $v \in L^2([0,T]; H_0^1(U))$, as functions of the form (9.3) are dense in this space. Hence in particular

$$\langle \mathbf{u}', v\rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v) \tag{9.6}$$

for each $v \in H_0^1(U)$ and a.e. $0 \leq t \leq T$.

Step three: In order to prove $\mathbf{u}(0) = g$, firstly note that from (9.5), integration by parts implies

$$\int_0^T -\langle \mathbf{v}', \mathbf{u}\rangle + B[\mathbf{u}, \mathbf{v}; t]\mathrm{d}t = \int_0^T (\mathbf{f}, \mathbf{v})\mathrm{d}t + (\mathbf{u}(0), \mathbf{v}(0)) \tag{9.7}$$

for each $\mathbf{v} \in C^1([0,T]; H_0^1(U))$ with $\mathbf{v}(T) = 0$; Similarly from (9.4) integration by parts shows

$$\int_0^T -\langle \mathbf{v}', \mathbf{u}_m\rangle + B[\mathbf{u}_m, \mathbf{v}; t]\mathrm{d}t = \int_0^T (\mathbf{f}, \mathbf{v})\mathrm{d}t + (\mathbf{u}_m(0), \mathbf{v}(0)) \tag{9.8}$$

Set $m = m_l$ and pass to weak limits we have

$$\int_0^T -\langle \mathbf{v}', \mathbf{u}\rangle + B[\mathbf{u}, \mathbf{v}; t]\mathrm{d}t = \int_0^T (\mathbf{f}, \mathbf{v})\mathrm{d}t + (g, \mathbf{v}(0))$$

As $\mathbf{v}(0)$ is arbitrary, together (9.7) and (9.8) we can conclude $\mathbf{u}(0) = g$. □

Theorem 9.3.4 (uniqueness of weak solution)**.** A weak solution of (9.1) is unique.

Proof. It suffices to check that the only weak solution of (9.1) with $\mathbf{f} \equiv g \equiv 0$ is $\mathbf{u} \equiv 0$. □

9.4. **Regularity.** In this section we discuss the regularity of weak solution $\mathbf{u}$ of (9.1). Our eventual goal is to prove that $\mathbf{u}$ is smooth, if the coefficients of the PDE, the boundary of the domain, etc. are smooth.

9.5. **Maximum principles.** In this section, we will establish maximum priciples of second-order parabolic equation in non-divergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

where a^{ij}, b^i, c are continous. The conclusions are quite similar to what we have established in the setting of elliptic equations.

Notation 9.5.1.

$$C_1^2(U_T) = \{u : U_T \to \mathbb{R} \mid u, D_x u, D_x^2 u, u_t \in C(U_T)\}$$

Theorem 9.5.1 (weak maximum priciple)**.** Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ and $c \equiv 0$ in U_T.

1. If $u_t + Lu \leq 0$ in U_T, then

$$\max_{\overline{U}_T} = \max_{\Gamma_T} u$$

2. If $u_t + Lu \geq 0$ in U_T, then

$$\min_{\overline{U}_T} = \min_{\Gamma_T} u$$

Theorem 9.5.2 (weak maximum priciple)**.** Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ and $c \geq 0$ in U_T.

1. If $u_t + Lu \leq 0$ in U_T, then

$$\max_{\overline{U}_T} = \max_{\Gamma_T} u^+$$

2. If $u_t + Lu \geq 0$ in U_T, then

$$\min_{\overline{U}_T} = -\max_{\Gamma_T} u^-$$

Theorem 9.5.3 (parabolic Harnack's inequality)**.** Assume $u \in C_1^1(U_T)$ with $u \geq 0$ solves $u_t + Lu = 0$ in U_T, and suppose $V \Subset U$ is connected. Then for each $0 < t_1 < t_2 \leq T$, there exists a constant $C(L, V, t_1, t_2)$ depending only on L, V, t_1 and t_2 such that

$$\sup_V u(\text{-}, t_1) \leq C \inf_V u(\text{-}, t_2)$$

Theorem 9.5.4 (strong maximum priciple)**.** Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ and $c \equiv 0$ in U_T, and we assume suppose U is connected.

1. If $u_t + Lu \leq 0$ in U_T, and u attains its maximum over $\overline{U}_T$ at a point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0};
2. If $u_t + Lu \geq 0$ in U_T, and u attains its minimum over $\overline{U}_T$ at a point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0}.

Theorem 9.5.5 (strong maximum priciple)**.** Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ and $c \geq 0$ in U_T, and we assume suppose U is connected.

1. If $u_t + Lu \leq 0$ in U_T, and u attains a non-negative maximum over $\overline{U}_T$ at a point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0};
2. If $u_t + Lu \geq 0$ in U_T, and u attains a non-positive minimum over $\overline{U}_T$ at a point $(x_0, t_0) \in U_T$, then u is constant on U_{t_0}.

Part 4. Appendix

Appendix A. L^p space

Let U be an open subset of $\mathbb{R}^n$.

A.1. First properties.

Definition A.1.1. For a measurable function $f : U \to \mathbb{R}^n$, if f satisfies

$$\int_U |f|^p \mathrm{d}x < \infty$$

then f is called p-th power integrable function, The set of all p-th power integrable functions on U is denoted by $L^p(U)$.

Definition A.1.2 (locally integrable)**.** For a measurable function $f : U \to \mathbb{R}^n$, if f satisfies

$$\int_K |f| \mathrm{d}x < +\infty$$

for all $K \Subset U$, then f is called locally integrable. The set of all locally integrable functions on U is denoted by $L^1_{loc}(U)$.

Remark A.1.1. By the same way we can define $L^p_{loc}(U)$. However it must be contained in $L^1_{loc}(U)$. Indeed, take $f \in L^p_{loc}(U)$, then for arbitrary $K \Subset U$ we have

$$\begin{aligned}\int_K |f| \mathrm{d}x &\le (\int_K |f|^p \mathrm{d}x)^{\frac{1}{p}} (\int_K \mathrm{d}x)^{\frac{1}{q}} \\ &< \infty\end{aligned}$$

where q such that $1/p + 1/q = 1$. In particular, since $L^p(U) \subset L^p_{loc}(U)$, we have

$$L^p(U) \subset L^1_{loc}(U)$$

for $1 \le p \le \infty$.

A.2. Convergence theorems.

Theorem A.2.1 (bounded convergence theorem)**.** Suppose that $\{f_n\}$ is a sequence of measure functions that are all bounded by constant M, are supported on a set E of finite measure, and $f_n(x) \to f(x)$ a.e. x as $n \to \infty$. Then f is measurable, bounded, supported in E for a.e. x, and

$$\lim_{n\to\infty} \int |f_n - f| = 0$$

In particular,

$$\lim_{n\to\infty} \int f_n = \int f$$

Lemma A.2.1 (Fatou)**.** Suppose $\{f_n\}$ is a sequence of measure functions with $f_n \ge 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then

$$\int f \le \liminf_{n\to\infty} \int f_n$$

Theorem A.2.2 (monotone convergence theorem)**.** Suppose $\{f_n\}$ is a increasing sequence of non-negative measurable function with $f_n \to f$ a.e. x, then

$$\lim_{n\to\infty} \int f_n = \int f$$

Theorem A.2.3 (dominant convergence theorem)**.** Suppose $\{f_n\}$ a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x. If $|f_n(x)| \leq g(x)$, where g is integrable, then

$$\lim_{n\to\infty} |f_n - f| = 0$$

In particular,

$$\lim_{n\to\infty} \int f_n = \int f$$

A.3. **Fubini theorem.**

Theorem A.3.1 (Fubini theorem)**.** Suppose $f(x, y)$ is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:

1. The slice f^y is integrable on $\mathbb{R}^{d_1}$;
2. The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)\mathrm{d}x$ is integrable on $\mathbb{R}^{d_2}$;
3. Furthermore,

$$\int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f(x, y)\mathrm{d}x)\mathrm{d}y = \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} f$$

Theorem A.3.2 (Tonelli theorem)**.** Suppose $f(x, y)$ is a non-negative measurable function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:

1. The slice f^y is integrable on $\mathbb{R}^{d_1}$;
2. The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)\mathrm{d}x$ is integrable on $\mathbb{R}^{d_2}$;
3. Furthermore,

$$\int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f(x, y)\mathrm{d}x)\mathrm{d}y = \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} f$$

Remark A.3.1. In practice, Tonelli theorem is often used in conjunction with Fubini's theorem. Indeed, suppose we are given a measurable function f on $\mathbb{R}^d$ and asked to compute $\int_{\mathbb{R}^d} f$. To justify the use of iterated integration, we first apply Tonelli theorem to $|f|$. Using it we can freely compute or estimate the iterated integrals of $|f|$. If we can show it's finite, thus f is integrable so we can use Fubini's theorem.

Appendix B. Hölder space

Let U be a bounded, open subset of $\mathbb{R}^n$.

B.1. Space $C^{0,\gamma}(\overline{U})$.

Definition B.1.1. Let $0 < \gamma \leq 1$. Then $C^{0,\gamma}(\overline{U})$ is the subset of $C(\overline{U})$ consisting of u such that

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup\{\frac{|u(x) - u(y)|}{|x-y|^\gamma} : x, y \in U, x \neq y\} < 0$$

where $[u]_{C^{0,\gamma}(\overline{U})}$ is called Hölder seminorm.

Definition B.1.2. For $u \in C^{0,\gamma}(\overline{U})$, its norm is defined as

$$\|u\|_{C^{0,\gamma}(\overline{U})} := \|u\|_{L^\infty(U)} + [u]_{C^{0,\gamma}(\overline{U})}$$

called γ-th Hölder norm.

Proposition B.1.1. Suppose $0 < \gamma \leq 1$, then $C^{0,\gamma}(\overline{U})$ is a Banach space.

Proof. Let $\{u_n\}$ be a Cauchy sequence of $C^{0,\gamma}(\overline{U})$, then it's also a Cauchy sequence of $C(\overline{U})$. Hence by completeness of $C(\overline{U})$, there exists $u \in C(\overline{U})$ such that $u = \lim_{n\to\infty} u_n$ in $C(\overline{U})$, that is u_n uniformly converges to u. Now we're going to show $u \in C^{0,\gamma}(\overline{U})$ and u_n converges to u in $C^{0,\gamma}(\overline{U})$.

Since Cauchy sequence is bounded, then there exists $M > 0$ such that $\|u_n\|_{C^{0,\gamma}(\overline{U})} \leq M$ for all $n \in \mathbb{N}$. Hence for any $x, y \in U$ with $x \neq y$

$$\frac{|u(x) - u(y)|}{|x-y|^\gamma} = \lim_{n\to\infty} \frac{|u_n(x) - u_n(y)|}{|x-y|^\gamma} \leq M$$

which implies $u \in C^{0,\gamma}(\overline{U})$.

In order to show desired convergence, fix $\varepsilon > 0$ and choose N such that $\|u_m - u_n\|_{C^{0,\gamma}(U)} < \varepsilon$ if $m, n \geq N$. Then for any $x, y \in U$ with $x \neq y$

$$\begin{aligned}\frac{|(u(x) - u_n(x)) - (u(y) - u_n(y))|}{|x-y|^\gamma} &= \lim_{n\to\infty} \frac{|(u_m(x) - u_n(x)) - (u_m(y) - u_n(y))|}{|x-y|^\gamma} \\ &\leq \limsup_{m\to\infty} [u_m - u_n]_{C^{0,\gamma}(U)} \\ &\leq \varepsilon\end{aligned}$$

provided $n \geq N$. Thus $[u - u_m]_{C^{0,\gamma}(\overline{U})} \leq \varepsilon$. Since ε is chosen arbitrarily and we already have uniform convenience, this shows $u_n \to u$ in $C^{0,\gamma}(\overline{U})$. □

Proposition B.1.2. Let $0 < \alpha < \beta \leq 1$ and $u \in C^{0,\beta}(\overline{U})$, then

$$[u]_{C^{0,\alpha}(\overline{U})} \leq [u]_{C^{0,\beta}(\overline{U})}$$

In particular, we have $C^{0,\beta}(\overline{U}) \subset C^{0,\alpha}(\overline{U})$

Proof. For any $x, y \in U$ with $x \neq y$, just note that

$$\begin{aligned}|u(x) - u(y)| &\leq [u]_{C^{0,\beta}(\overline{U})}|x-y|^\beta \\ &\leq [u]_{C^{0,\beta}(\overline{U})}|x-y|^{\beta-\alpha}|x-y|^\alpha \\ &\leq [u]_{C^{0,\beta}(\overline{U})}(\operatorname{diam} U)^{\beta-\alpha}|x-y|^\alpha\end{aligned}$$

□

Proposition B.1.3. If U is convex, then $C^1(\overline{U}) \subset C^{0,1}(\overline{U})$.

Proof. For $u \in C^1(\overline{U})$, it suffices to show $[u]_{C^{0,1}(\overline{U})} < \infty$. Note that for arbitrary $x, y \in U$, one has

$$\frac{|u(x) - u(y)|}{|x-y|} = |Du(c)|$$

for some $c \in U$, since U is convex, we can use mean value theorem. Thus we have

$$\frac{|u(x) - u(y)|}{|x-y|} \leq \|Du\|_{C(\overline{U})} < \infty$$

Taking a supremum we obtain the desired result. □

B.2. **Space $C^{k,\gamma}(\overline{U})$.**

Definition B.2.1. Let $0 < \gamma \leq 1$. Then $C^{k,\gamma}(\overline{U})$ is the subset of $C^k(\overline{U})$ consisting of u such that the following norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} := \sum_{|\alpha|\leq k}\|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|\leq k}[D^\alpha u]_{C^{0,\gamma}(\overline{U})}$$

is finite.

Proposition B.2.1. Suppose $0 < \gamma \leq 1$, then $C^{k,\gamma}(\overline{U})$ is a Banach space.

Proposition B.2.2. We have the following inclusions:

1. For $0 < \alpha < \beta \leq 1$, $C^{k,\beta}(\overline{U}) \subset C^{k,\alpha}(\overline{U})$;
2. If U is convex, then $C^{k+1}(\overline{U}) \subset C^{k,1}(\overline{U})$.

Appendix C. Approximation

C.1. **Convolution in $\mathbb{R}^n$.** For two functions f, g on $\mathbb{R}^n$, we can formally define

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y$$

which is called convolution of f and g. If such integral exists, there are some basic properties:

1. $f * g = g * f$;
2. $(f * g) * h = f * (g * h)$;
3. $\mathrm{supp}(f * g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$.

Let's see some cases in which $f * g(x)$ is well defined for almost all x.

Theorem C.1.1. If $f, g \in L^1(\mathbb{R}^n)$. Then for almost all x, the function $f(x-y)g(y)$ is integrate in y, and

$$\int_{\mathbb{R}^n} |f * g(x)|\mathrm{d}x \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}$$

Proof. Note that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2n}$, and by Tonnelli theorem:

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} |f(x-y)g(y)|\mathrm{d}x\mathrm{d}y &= \int (\int |f(x-y)||g(y)|\mathrm{d}x)\mathrm{d}y \\
&= (\int |f(x)|\mathrm{d}x)(\int |g(y)|\mathrm{d}y) \\
&< \infty
\end{aligned}$$

Then by Fubini theorem, $f(x-y)g(y)$ is integrable in y for almost all x, that is $f * g(x)$ is defined for almost all $x \in \mathbb{R}^n$. Furthermore,

$$\begin{aligned}
\int |f * g(x)|\mathrm{d}x &= \int |\int f(x-y)g(y)\mathrm{d}y|\mathrm{d}x \\
&\leq \int\int |f(x-y)||g(y)|\mathrm{d}y\mathrm{d}x \\
&= \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}
\end{aligned}$$

$\square$

Remark C.1.1. Convolution product turns the Banach space $L^1(\mathbb{R}^n)$ into a communicative Banach algebra.

Remark C.1.2. More common usages of convolution: Suppose $K(x) \in L^1(\mathbb{R}^n)$. Then the linear mapping

$$f \mapsto K * f$$

is a bounded map on $L^1(\mathbb{R}^n)$ with operator norm $\leq \|K\|_{L^1(\mathbb{R}^n)}$. Such K is called a convolution kernel.

Theorem C.1.2. If $K \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$. Then for almost all x, the function $K(x-y)f(y)$ is p-th power integrate in y, and

$$\|K * f\|_{L^p(\mathbb{R}^n)} \leq \|K\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

Proof. For $p = \infty$, note that

$$\begin{aligned}
\int_{\mathbb{R}^n} |K(x-y)f(y)| \mathrm{d}y &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \int |K(y)| \mathrm{d}y \\
&= \|f\|_{L^\infty(\mathbb{R}^n)} \|K\|_{L^1(\mathbb{R}^n)} \\
&< \infty
\end{aligned}$$

thus $K * f(x)$ exists for almost all x, and

$$\begin{aligned}
|\int K(x-y)f(y)\mathrm{d}y| &\leq \int |K(x-y)| \mathrm{d}y \|f\|_{L^\infty(\mathbb{R}^n)} \\
&= \|K\|_{L^1(\mathbb{R}^n)} \|f\|_{L^\infty(\mathbb{R}^n)}
\end{aligned}$$

which implies

$$\|K * f\|_{L^\infty(\mathbb{R}^n)} \leq \|K\|_{L^1(\mathbb{R}^n)} \|f\|_{L^\infty(\mathbb{R}^n)}$$

For $1 < p < \infty$, and choose q such that $1/p + 1/q = 1$. By Tonelli theorem,

$$\begin{aligned}
\int |K(x-y)f(y)|^p \mathrm{d}x\mathrm{d}y &= \int (\int |K(x-y)f(y)| \mathrm{d}y)^p \mathrm{d}x \\
&= \int (\int |K(x-y)^{\frac{1}{q}} K^{\frac{1}{p}}(x-y) f(y)| \mathrm{d}y)^p \mathrm{d}x \\
&\leq \|K\|^{\frac{p}{q}}_{L^1(\mathbb{R}^n)} \int (\int |K(x-y) f^p(y)| \mathrm{d}y) \mathrm{d}x \\
&= \|K\|^{\frac{p}{q}}_{L^1(\mathbb{R}^n)} \int |K(y)| \mathrm{d}y \int |f^p(x-y)| \mathrm{d}x \\
&= \|K\|^{\frac{p}{q}+1}_{L^1(\mathbb{R}^n)} \|f\|^p_{L^p(\mathbb{R}^n)} \\
&< \infty
\end{aligned}$$

Then by Fubini theorem, $|K(x-y)f(y)|^p$ is integrable in y for almost all x, that is $f * g(x) \in L^p(\mathbb{R}^n)$ is defined for almost all $x \in \mathbb{R}^n$. Furthermore,

$$\begin{aligned}
\|K * f\|^p_{L^p(\mathbb{R}^n)} &\leq \int (\int |K(x-y)f(y)| \mathrm{d}y)^p \mathrm{d}x \\
&\leq \int (\int |K(x-y)^{\frac{1}{q}} K^{\frac{1}{p}}(x-y) f(y)| \mathrm{d}y)^p \mathrm{d}x \\
&\leq \|K\|^{\frac{p}{q}}_{L^1(\mathbb{R}^n)} \int \int |K(x-y) f^p(y)| \mathrm{d}y\mathrm{d}x \\
&= \|K\|^{\frac{p}{q}}_{L^1(\mathbb{R}^n)} \int |K(y)| \mathrm{d}y \int |f^p(x-y)| \mathrm{d}x \\
&= \|K\|^{\frac{p}{q}+1}_{L^1(\mathbb{R}^n)} \|f\|^p_{L^p(\mathbb{R}^n)} \\
&= (\|K\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)})^p
\end{aligned}$$

$\square$

Remark C.1.3. In fact, more general we have if $K \in L^p(\mathbb{R}^n)$, $f \in L^q(\mathbb{R}^n)$ such that $1/p+1/q \geq 1$, then $K*f \in L^r(\mathbb{R}^n)$, where r satisfies $1/p+1/q = 1+1/r$, and

$$\|K * f\|_{L^r(\mathbb{R}^n)} \leq \|K\|_{L^p(\mathbb{R}^n)}\|f\|_{L^q(\mathbb{R}^n)}$$

C.2. **Approximation to the identity.**

Definition C.2.1. If $K \in L^1(\mathbb{R}^n)$, define

$$K_\varepsilon(x) := \frac{1}{\varepsilon^n}K(\frac{x}{\varepsilon})$$

Remark C.2.1. Here are two remarks about K_ε:

1. By change of variables, we have

$$\int K_\varepsilon(x)\mathrm{d}x = \frac{1}{\varepsilon^n}\int K(\frac{x}{\varepsilon}) = \int K(x)\mathrm{d}x$$

2. If $\delta > 0$, then

$$\int_{|x|>\delta} |K_\varepsilon(x)|\mathrm{d}x = \int_{|x|>\frac{\delta}{\varepsilon}} |K(x)|\mathrm{d}x$$

so for any fixed $\delta > 0$, we have

$$\lim_{\varepsilon\to 0}\int_{|x|>\delta} |K_\varepsilon(x)|\mathrm{d}x = 0$$

Theorem C.2.1 (approximation to the identity)**.** If $K \in L^1(\mathbb{R}^n)$ and $\int K(x)\mathrm{d}x = 1$. For $f \in L^p(\mathbb{R}^n)$ and $1 \leq p < \infty$, then

$$\lim_{\varepsilon\to 0}\|K_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)} = 0$$

Proof. Note that

$$\begin{aligned}
\|K_\varepsilon * f - f\|^p_{L^p(\mathbb{R}^n)} &= \int |\int \{K_\varepsilon(x-y)f(y)\}\mathrm{d}y - f(x)|^p\mathrm{d}x \\
&= \int |\int \{K_\varepsilon(y)(f(x-y) - f(x))\}\mathrm{d}y|^p\mathrm{d}x \\
&\leq \int\int |K_\varepsilon(y)|^p|f(x-y) - f(x)|^p\mathrm{d}y\mathrm{d}x \\
&\leq \|K_\varepsilon\|^{\frac{p}{q}}_{L^1(\mathbb{R}^n)}\int\int |K_\varepsilon(y)||f(x-y) - f(x)|^p\mathrm{d}x\mathrm{d}y
\end{aligned}$$

Fix $\eta > 0$, we can find $\delta > 0$ such that

$$\int |f(x-y) - f(x)|^p\mathrm{d}x \leq \frac{\eta}{2}$$

when $|y|<\delta$. Thus we write

$$\begin{aligned}\int\int |K_\varepsilon(y)||f(x-y)-f(x)|^p\mathrm{d}y\mathrm{d}x &= \int_{|y|<\delta}|K_\varepsilon(y)|\int|f(x-y)-f(x)|^p\mathrm{d}x\mathrm{d}y\\ &\quad+\int_{|y|\geq\delta}|K_\varepsilon(y)|\int|f(x-y)-f(x)|^p\mathrm{d}x\mathrm{d}y\\ &\leq \frac{\eta}{2}+2\|f\|_{L^p(\mathbb{R}^n)}\int_{|y|\geq\delta}K_\varepsilon(y)\mathrm{d}y\end{aligned}$$

Then we can find ε' such that for any $0<\varepsilon<\varepsilon'$ we have

$$2\|f\|_{L^p(\mathbb{R}^n)}\int_{|y|\geq\delta}K_\varepsilon(y)\mathrm{d}y<\frac{\eta}{2}$$

Thus for any $0<\varepsilon<\varepsilon'$, we have

$$\|K_\varepsilon * f-f\|^p_{L^p(\mathbb{R}^n)}\leq\eta\|K_\varepsilon\|^{\frac{p}{q}}_{L^1(\mathbb{R}^n)}$$

This completes the proof. □

C.3. **Approximation of $L^p(\mathbb{R}^n)$.** In this section, we take a special convolution kernel, that is to take $K(x)$ to be some smooth function with compact support, then we can see some useful results in approximation of L^p.

Definition C.3.1 (mollifier)**.** Define $\phi\in C_c^\infty(\mathbb{R}^n)$ by

$$\phi(x)=\begin{cases}C\exp(\frac{1}{|x|^2-1}), & |x|<1\\ 0, & |x|\geq 1\end{cases}$$

where $C>0$ such that $\int_{\mathbb{R}^n}\phi(x)\mathrm{d}x=1$.

Definition C.3.2 (standard sequence of mollifier)**.** For $\varepsilon>0$, the standard sequence of mollifiers on $\mathbb{R}^n$ is defined by

$$\phi_\varepsilon(x):=\frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$$

Remark C.3.1*.* It's clear $\int_{\mathbb{R}^n}\eta_\varepsilon(x)\mathrm{d}x=1$ and $\mathrm{supp}(\eta_\varepsilon)\subset\overline{B(0,\varepsilon)}$.

Now take our convolution kernel $K(x)=\phi(x)$ we defined above, for any $f\in L^p(\mathbb{R}^n)$ where $1\leq p<\infty$, we have

$$\lim_{\varepsilon\to 0}\|\phi_\varepsilon * f-f\|_{L^p(\mathbb{R}^n)}=0$$

that is a sequence converging to f in $L^p(\mathbb{R}^n)$. Furthermore, we have

Proposition C.3.1. For $f\in L^p(\mathbb{R}^n)$ with $1\leq p\leq\infty$ and $\varepsilon>0$, $\phi_\varepsilon * f\in C^\infty(\mathbb{R}^n)$.

Thus we obtain the first approximation:

Corollary C.3.1. For any $1\leq p<\infty$, $C^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Furthermore, we can do it better:

Corollary C.3.2. For any $1 \leq p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Proof. For any $f \in L^p(\mathbb{R}^n)$, there exists a compact set $K \subset \mathbb{R}^n$ such that given arbitrary $\delta > 0$, we have

$$\int_{\mathbb{R}^n - K} |f|^p \mathrm{d}x < \frac{\delta}{2}$$

Thus consider $f\chi_K \in L^p(\mathbb{R}^n)$, with compact support. It's clear $\phi_\varepsilon * f\chi_K$ still has compact support, and there exists ε' such that for any $0 < \varepsilon < \varepsilon'$ we have

$$\|\phi_\varepsilon * f\chi_K - f\chi_K\|_{L^p(\mathbb{R}^n)} < \frac{\delta}{2}$$

Thus

$$\begin{aligned}\|\phi_\varepsilon * f\chi_K - f\|_{L^p(\mathbb{R}^n)} &\leq \|\phi_\varepsilon * f\chi_K - f\chi_K\|_{L^p(\mathbb{R}^n)} + \|f\chi_K - f\|_{L^p(\mathbb{R}^n)} \\ &\leq \delta\end{aligned}$$

□

Remark C.3.2. Note that this theorem fails for $p = \infty$, there exists $f \in L^\infty(\mathbb{R}^n)$ can not be approximated by any continous function(no matter it's compactly supported or not) in L^∞-norm. For example, let's take $n = 1$ and consider

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Then any continous function with $\|f - g\|_{L^\infty} < \frac{1}{3}$ must have $g(x) < f(x) + \frac{1}{3}$ for all $x < 0$. By continuity, we have $g(0) \leq \frac{1}{3}$, contradicting $g(0) > f(0) - \frac{1}{3} = \frac{2}{3}$.

Remark C.3.3. There is another way to show $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$: Firstly you need to use Lusin theorem to show continous functions with compact support is dense in $L^p(\mathbb{R}^n)$, then use convolution to mollifier these continous functions. All in all, you do need convolution.

C.4. **Approximation of $L^p_{loc}(\mathbb{R}^n)$.** For any $f \in L^p_{loc}(\mathbb{R}^n)$, consider

$$\begin{aligned}\int_{\mathbb{R}^n} |\phi_\varepsilon(x-y)f(y)|^p \mathrm{d}y &= \int_{B(0,\varepsilon)} |\phi_\varepsilon(y)f(x-y)|^p \mathrm{d}y \\ &= \int_{B(0,1)} |\phi(z)f(x-\varepsilon z)|^p \mathrm{d}z \\ &\leq \int_{B(0,1)} |f(x-\varepsilon z)|^p \mathrm{d}z \\ &< \infty\end{aligned}$$

Thus $f^\varepsilon := \phi_\varepsilon * f$ is well defined for almost all x. It's clear f^ε is also a smooth function, so we desired some density for smooth functions in $L^p_{loc}(\mathbb{R}^n)$, as what we have done in $L^p(\mathbb{R}^n)$.

Theorem C.4.1. For any $1 \leq p < \infty$, $C^\infty(\mathbb{R}^n)$ is dense in $L^p_{loc}(\mathbb{R}^n)$.

Proof. For $f \in L^p_{loc}(\mathbb{R}^n), 1 \leq p < \infty$, we claim $f^\varepsilon \in C^\infty(\mathbb{R}^n)$ converges to f in $L^p_{loc}(\mathbb{R}^n)$. Indeed, for arbitrary compact set V, choose another compact set W such that $V \subset W$. Then for $x \in V$, we have

$$\begin{aligned}|f^\varepsilon(x)| &\leq \int_{B(0,1)} \phi(z)^{1-\frac{1}{p}} \phi(z)^{\frac{1}{p}} |f(x-\varepsilon z)| \mathrm{d}z \\ &= (\int_{B(0,1)} \phi(z)\mathrm{d}z)^{1-\frac{1}{p}} (\int_{B(0,1)} \phi(z)|f(x-\varepsilon z)|^p \mathrm{d}z)^{\frac{1}{p}} \\ &= (\int_{B(0,1)} \phi(z)|f(x-\varepsilon z)|^p \mathrm{d}z)^{\frac{1}{p}}\end{aligned}$$

Hence for $1 \leq p < \infty$ and sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\int_V |f^\varepsilon(x)|^p \mathrm{d}x &= \int_V (\int_{B(0,1)} \phi(z)|f(x-\varepsilon z)|^p \mathrm{d}z)\mathrm{d}x \\ &\leq \int_{B(0,1)} \phi(z)(\int_V |f(x-\varepsilon z)^p \mathrm{d}x)\mathrm{d}z \\ &\leq \int_W |f(y)|^p \mathrm{d}y\end{aligned}$$

that is $\|f^\varepsilon\|_{L^p(V)} \leq \|f\|_{L^p(W)}$ for sufficiently small ε.

Now fix $\delta > 0$, since $f \in L^p(W)$, there exists $g \in C(W)$ such that

$$\|f - g\|_{L^p(W)} \leq \frac{\delta}{3}$$

which implies

$$\|f^\varepsilon - g^\varepsilon\|_{L^p(V)} \leq \frac{\delta}{3}$$

Consequently

$$\begin{aligned}\|f^\varepsilon - f\|_{L^p(V)} &\leq \|f^\varepsilon - g^\varepsilon\|_{L^p(V)} + \|g^\varepsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)} \\ &\leq \frac{2\delta}{3} + \|g^\varepsilon - g\|_{L^p(V)}\end{aligned}$$

Since $g^\varepsilon \to g$ in $L^p(V)$, we can find ε' such that for any $0 < \varepsilon < \varepsilon'$, we have

$$\|g^\varepsilon - g\|_{L^p(V)} < \frac{\delta}{3}$$

This completes the proof. □

Remark C.4.1. Unfortunately, $C^\infty_c(\mathbb{R}^n)$ is not dense in $L^p_{loc}(\mathbb{R}^n)$.

C.5. **Approximation in open subset of $\mathbb{R}^n$.** Now we assume U is an open subset of $\mathbb{R}^n$, we also want to use smooth function to approximation function $f \in L^p_{loc}(U)$. In this case

$$f^\varepsilon(x) = \int_{B(0,1)} \phi(z) f(x - \varepsilon z)\mathrm{d}z$$

can't be defined on the whole U, since $x - \varepsilon z$ may not in U for some $x \in U$. Consider

$$U_\varepsilon := \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$$

Then f^ε is well-defined on U_ε, and it's smooth. Then by same proof of Theorem C.4.1, we can show $f^\varepsilon \to f$ in $L^p_{loc}(U)$, since for arbitrary $V \Subset U$, we can choose ε sufficiently small such that $V \subset U_\varepsilon$.

References

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